In many applications, it is desirable to have highly directional beams with the smallest possible spot size at the source. However, because of an inverse (Fourier-transform) relationship between the spot size of the beam and its far-field angular spread [1], it is only possible to trade one off for the other. To characterize the beam quality, a dimensionless quantity, known as the $M^2$ factor, is often used in practice [2]. Mathematically, this factor is related to the phase-space product of the root-mean-square (rms) beam width and the rms value of the far-field angle [3–7] and is defined such that $M^2=1$ for an ideal Gaussian beam whose beam waist is located at the source plane.

Even though $M^2$ factor was originally introduced in the context of continuous-wave (cw) beams, it is often used to characterize the quality of pulsed beams as well. In the case of relatively long pulses such as those emitted by Q-switched lasers, a time-dependent $M^2$ factor is sometimes employed [4–6]. Yet, such an approach is both fundamentally unsound and impractical in the case of femtosecond pulses emitted by modern mode-locked lasers [8]. For ultrashort pulses lasting for only a few optical cycles, the spectral bandwidth becomes comparable in magnitude to the carrier frequency of the pulse [9]. Although the properties of such pulsed beams, propagating in free space or linear dispersive media, have been extensively investigated in recent years [10–16], a rather subtle question of how to define the corresponding $M^2$ factor has so far not been addressed. A related basic question is: What is the radial intensity profile associated with the best-quality ultrashort pulsed beam?

In this Letter, we propose a definition for the phase-space quality factor of ultrashort pulsed beams. We show that the magnitude of such a pulse beam quality factor $M^2$ has a minimum value that depends on the characteristics of the pulse source spectrum. We also demonstrate that the radial intensity profile of the best-quality pulsed beam can significantly deviate from a Gaussian.

In the following, it is convenient to work in the space-frequency representation by introducing a spectral decomposition of the optical field $V(\mathbf{r}, t)$ of the pulse viz., $U^2(\mathbf{r}, t) = \int_0^\infty dt V(\mathbf{r}, t) \exp(i\omega t)$. The rms width of the pulsed beam at the source plane can then be defined as

$$\langle \rho^2 \rangle = \frac{\int_0^\infty d\omega \int d^2 \rho e^2 |U(\mathbf{r}, \omega)|^2}{\int_0^\infty d\omega \int d^2 \rho |U(\mathbf{r}, \omega)|^2},$$

where $|U(\mathbf{r}, \omega)|^2$ is the density of the energy spectrum $S(\omega)$, which was predicted to be a directly measurable quantity for femtosecond laser pulses [17]. For fully coherent pulsed beams, $S(\omega)$ is defined by the relation

$$S(\omega) = \int d^2 \rho |U(\mathbf{r}, \omega)|^2.$$

The rms value of the far-field angle can be defined in a similar manner using the concept of the radiant intensity. More precisely,

$$\langle s_\theta^2 \rangle = \frac{\int_0^\infty d\omega \int d^2 s_\theta s_\phi^2 J(k s_\perp, \omega)}{\int_0^\infty d\omega \int d^2 s_\theta J(k s_\perp, \omega)},$$

where the radiant intensity, $J(k s_\perp, \omega)$, is given by [1]

$$J(k s_\perp, \omega) = (2\pi)^2 \cos^2 \theta \bar{U}(k s_\perp, \omega)^2.$$

Here, $k=\omega/c$ is the propagation constant, $s_\perp$ is a two-dimensional (2D) vector projection onto the source plane of a [three-dimensional (3D)] unit vector $\mathbf{s}$ pointing from the source to the far zone, and $\bar{U}(k s_\perp, \omega)$ is the 2D spatial Fourier transform of $U(\mathbf{r}, \omega)$.

In most practical situations, one deals with paraxial sources whose angular distribution peaks sharply along the $z$ direction such that $\cos \theta \approx 1$ and $|s_\perp| \approx \sin \theta \approx \theta$. Hence, we can extend the limits of in-
integration over \( k s \) to cover the entire 2D Fourier plane. It then follows from Eqs. (3) and (4) that the angular spread of a pulse beam generated by a paraxial source is given by

\[
\langle s^2 \rangle = \frac{\int_0^\infty \! d\omega \int d^2(k s) |\hat{U}(k s, \omega)|^2}{\int_0^\infty \! d\omega \int d^2(k s) |\hat{U}(k s, \omega)|^2},
\]

(5)

To obtain a lower bound for the phase-space product of pulsed beams, we generalize the approach of [18] by considering the following functional:

\[
\Phi(\alpha) = \frac{1}{W} \int_0^\infty \! d\omega \int d^2 \rho(f \cdot f) \geq 0,
\]

(6)

where the vector \( f \) is defined as

\[
f = \rho U(\rho, \omega) + \alpha(\omega) \nabla U(\rho, \omega),
\]

(7)

\( \alpha \) is any real function of \( \omega \), and the total energy \( W \) of the pulse is expressed as

\[
W = \int_0^\infty \! d\omega \int d^2 \rho |U|^2 = \int_0^\infty \! d\omega \int d^2(k s) |\hat{U}|^2.
\]

(8)

On observing that

\[
\int d^2 \rho \nabla U^* \nabla U = \int d^2(k s) k^2 s^2 |\hat{U}|^2,
\]

(9)

which follows from the properties of Fourier transforms, we can cast the inequality in Eq. (6) into the form

\[
\langle \rho^2 \rangle - 2 \alpha_0 F_s(\omega_0) + \alpha_0^2 k^2 s^2 \geq 0,
\]

(10)

provided that \( \alpha \) obeys the scaling relation \( \alpha(\omega) = \alpha_0(\omega_i/\omega) \). In Eq. (10), \( F_s \) represents a spectral form factor, defined as

\[
F_s(\omega_0) = \frac{\int_0^\infty \! d\omega \left( \frac{\omega_0}{\omega} \right) S(\omega)}{\int_0^\infty \! d\omega S(\omega)},
\]

(11)

where \( \omega_0 \) is the frequency at which the source spectrum attains maximum.

We now introduce the \( M^2 \) factor in the form of a phase-space product as

\[
M^2 = k_0 \sqrt{\langle s^2 \rangle / \langle \rho^2 \rangle},
\]

(12)

and we propose to use it for assessing the phase-space quality of any pulsed beam. The advantage of our definition over the previously introduced time-resolved one [6] is that the former represents a time-independent scalar quantity characterizing the whole pulse. It follows at once from Eq. (10) that there exists a lower bound on the magnitude of \( M^2 \) such that

\[
M^2 \geq F_s(\omega_0).
\]

(13)

Inequality (13) is a key result of this Letter. It shows that the quality factor of a pulsed beam, in general, depends on the spectral characteristics of the corresponding source. As required, the new inequality reduces to the standard result \( M^2 \geq 1 \) for cw or quasi-cw sources whose bandwidth is narrow enough, \( S(\omega) \propto \delta(\omega - \omega_0) \).

For ultrashort pulses with a relatively wide spectrum, the minimum value of \( M^2 \) exceeds 1. Consider, for example, a Gaussian pulse whose spectrum is of the form \( S(\omega) \propto \exp\left[-(\omega - \omega_0)^2 / \delta^2\right] \), where \( \delta \) is a measure of the spectral bandwidth. It is seen in Fig. 1 that \( F_s \) monotonically increases as a function of the ratio \( \delta / \omega_0 \), exceeding 2 for \( \delta / \omega_0 > 0.7 \). Clearly, \( M^2 \) close to 1 should not be expected for pulsed beams whose spectral bandwidth becomes comparable to their carrier frequencies. It should be stressed that the choice of a Gaussian spectrum becomes inappropriate as \( \delta / \omega_0 \rightarrow 1 \) because \( S(\omega) \) does not vanish at \( \omega = 0 \), as required of any physically realizable spectrum.

To model more accurately the ultrashort pulse spectrum, we consider a realistic femtosecond pulse source [9]. The energy spectrum of any such source should exhibit (a) an infrared cutoff and the parameter \( \alpha = q/\omega_0 \) and (b) a degree of asymmetry with a high-frequency tail, and (c) a well-defined peak at a certain frequency. We propose the following phenomenological model for such a spectrum:

\[
S(\omega) \propto \begin{cases} (\omega - \omega_0)^q e^{-\omega} & \text{if } \omega \geq \omega_s, \\ 0 & \text{if } \omega \leq \omega_s, \end{cases}
\]

(14)

where \( \omega_0 \) is the infrared cutoff frequency and the parameter \( \alpha = q/\omega_0 \) is chosen such that the spectrum peaks at \( \omega_0 \). The real positive exponent \( q \) and the ratio \( \omega_0 / \omega_s \) describe the asymmetry of the spectrum. In Fig. 2 we display pulse spectra (top) for \( q = 1 \), and the corresponding form factors (bottom) as functions of \( q \) for four values of the ratio \( \omega_0 / \omega_s \). When this ratio is relatively large, \( F_s \) exceeds 2. Values closer to 1 are realized for \( \omega_0 / \omega_s < 10 \), and \( F_s \) becomes smaller than 1 when \( \omega_0 / \omega_s \) is below 6.

An important question is under what conditions \( M^2 \) of a pulsed beam attains its minimum value \( F_s \).
The field distribution of such a beam can be determined from Eqs. (6) and (7) by noting that $f$ should be set to zero, or

$$\rho U(\rho_0, \omega) + \alpha(\omega) \nabla U(\rho_0, \omega) = 0.$$  \hspace{1cm} (15)

A straightforward integration of this equation yields

$$U(\rho_0, \omega) = A(\omega) \exp \left[-\frac{\rho^2}{2\sigma^2(\omega)}\right],$$  \hspace{1cm} (16)

where $A(\omega)$ is the spectral density of the pulsed beam on the axis, and we have introduced the (frequency-dependent) beam width $\sigma(\omega)$ by the expression

$$\sigma(\omega) = \frac{\sigma_0(\omega_0/\omega)^{1/2}}{}.$$  \hspace{1cm} (17)

As is seen from Eq. (16), the field $U(\rho_0, \omega)$ does not factorize into spectral and spatial parts for the best phase-space quality pulsed beam. To study the spatial distribution of the energy in the pulsed beam, we determine its radial intensity profile, defined as

$$I(\rho) = \int_0^\infty d\omega |U(\rho_0, \omega)|^2.$$  \hspace{1cm} (18)

Substituting from Eq. (16) into Eq. (18) and eliminating $A$ in favor of the measurable energy spectrum, we obtain

$$I(\rho) = \frac{1}{4\pi} \int_0^\infty \frac{d\omega}{\sigma^2(\omega)} \exp \left[-\frac{\rho^2}{2\sigma^2(\omega)}\right].$$  \hspace{1cm} (19)

In Fig. 3, we show the radial intensity profile $I(\rho)$ for two values of $\omega_0/\omega$. The dashed curve shows a Gaussian profile of width $\sigma_0$ for comparison.

Two values of $\omega_0/\omega$ for ultrashort pulses with the spectrum given in Eq. (14). A Gaussian profile for a quasi-monochromatic beam centered at $\omega_0$ is also presented for comparison. Clearly, the beam profile deviates considerably from a Gaussian shape for such broadband pulse spectra.

In conclusion, we have shown that the condition $M^2 \geq 1$ does not hold for optical beams consisting of ultrashort pulse trains. Rather, the minimum value of $M^2$ depends on the pulse spectrum. It exceeds 1 in most cases of practical interest, but $M^2 < 1$ is possible for certain pulse spectra. We also show that the radial beam profile of the best phase-space quality pulse can significantly deviate from a Gaussian shape.

References and Note

2. An extensive list of references on beam quality (up to 1998) can be found at http://www.stanford.edu/~siegman/beam_quality_ref_list.html.