A problem of continuing interest to scientists working in the areas of nanoscience, plasmonics, and metamaterials is the achievement of full control over the optical energy flow on a nanoscale. The need for such a control is dictated by the Internet-traffic demands and information processing challenges that need to be handled in the near future. It is expected that some of these challenges will be met by nonlinear plasmonic waveguides in which digital signals in the form of dielectrics and metals. It turns out that nonlinearities cause drastic modification of the SPP energy spectrum; these features have to be thoroughly understood using a suitable nonlinear theory prior to being beneficially employed in practice.

During the past three decades, ample attention has been devoted to deriving nonlinear dispersion equations for the SPP modes of metal–dielectric interfaces and linear metallic slabs surrounded by different types of nonlinear media. It is rather surprising that, to the best of our knowledge, no analytical expression for the SPP dispersion relation has been obtained using exact field decomposition of TM waves. Our approach generalizes the known linear dispersion relations prior to being beneficially employed in practice.

We derive an exact dispersion relation for the surface plasmon polaritons of a nonlinear plasmonic waveguide using exact field decomposition of TM waves. Our approach generalizes the known linear dispersion relations to the case of a medium nonlinearity of the form \( \varepsilon_{nl} = \varepsilon_L + \alpha |E|^n \). We apply the unique dispersion relation to a plasmonic waveguide with a Kerr-type nonlinearity \((n = 1)\) and show that it enables backward-propagating modes. It also introduces critical points in the energy spectrum of surface plasmon polaritons that result in enhanced interaction of nonlinear modes with each other and external electromagnetic fields.
where we marked the integration constant $C$ with the subscript ± to stress that its value depends on the symmetry of the SPP mode, determined by the symmetry of the transverse electric field with respect to the reflection in the plane $x = 0$. If we denote the amplitude of the electric field in this plane by $E_0$, then the constant may be written as

$$C_{\pm} = [\varepsilon_0 + n \varepsilon_L - \varepsilon_0 (2 - \varepsilon_0 \vartheta^2)(n + 1)\sigma_{\pm}] E_0^2,$$

(3) in this case provide

$$E_{z1} = \sqrt{[1 - \varepsilon_L \vartheta^2] E_z^2_{z1} + [1 - (2 - \varepsilon_L \vartheta^2)\sigma_{\pm}] E_0^2},$$

while Eq. (5) yields the relation

$$\left(\frac{\tilde{E}_1}{E_0}\right)^2 = \frac{1 - \sigma_{\pm} (2 - \varepsilon_L \vartheta^2)}{1 - (2 - \varepsilon_L \vartheta^2)/(1 + \eta_L^2)},$$

with $\eta_L = (\varepsilon_L/|\varepsilon|_2)/k_2/\beta$. Using these expressions in Eq. (4) and introducing a new parameter $q = \sqrt{1 - \varepsilon_L \vartheta^2}$, we come up with the expected result \cite{36,24-26}:

$$\tanh(q \beta h) = (\eta_L/q)^{\pm 1}.$$

For the purpose of illustrating the SPP dispersion peculiarities in nonlinear plasmonic waveguides, we focus on the Kerr nonlinearity for which Eqs. (2) and (5) can be solved analytically. From Eq. (2), we find that

$$E_{z1} = \left(\frac{b}{\alpha a} + \sqrt{\frac{b^2}{\alpha a} + ac_{\pm}} - E_{z1}^2\right)^{1/2},$$

where $a = 2 + 4\vartheta^2 \alpha E_{z1}^2$, $b = 4(1 - \varepsilon_L \vartheta^2)\alpha E_{z1}^2 - 2\varepsilon_L$, and $c_{\pm} = \alpha c_{\pm} + 4\varepsilon_L (2 - \varepsilon_L \vartheta^2)\alpha E_{z1}^2$. The roots of Eq. (5) are given by Ferrari’s formula. \cite{25} We assume that the plasmonic waveguide is characterized by the parameters $\varepsilon_L = 2.25$, $\varepsilon_2 = 1 - \omega_0^2/\omega^2$, and $\omega_0 = 1.36 \times 10^{16}$ Hz, and consider two values of the nonlinear coefficient $\alpha = \pm 2 \times 10^{-16}$ m$^2$ V$^{-2}$, with the plus and minus signs corresponding to self-focusing and self-defocusing nonlinear media, respectively.

Figure 2 shows dispersion relations $\omega_{\text{SPP}}(\beta)$ in the self-focusing case ($\alpha > 0$) in the case of two nonlinear

FIG. 1. (Color online) Schematic dependence of the electric field components $E_x$ and $E_z$ on the transverse coordinate $x$ for (a) symmetric and (b) antisymmetric SPP modes of a nonlinear plasmonic waveguide, formed by a nonlinear dielectric layer of thickness $2h$ and permittivity $\varepsilon_1$ sandwiched between two metallic layers of permittivity $\varepsilon_2$.

FIG. 2. (Color online) Dispersion curves of SPPs for three values of $E_0$ (electric field at mode center at $x = 0$) in the case of 100- and 250-nm-thick nonlinear plasmonic waveguides with a self-focusing Kerr nonlinearity ($\alpha > 0$). Thin circles marked by the letters S (symmetric mode) and A (antisymmetric mode) indicate specific points on dispersion curves corresponding to waves moving forward (f) and backward (b); two critical points are marked as $A_c$ and $S_c$. The light line in the nonlinear medium with $\alpha = 0$ is shown as dotted. For material parameters, refer to the text.

113409-2
waveguides of thicknesses 100 and 250 nm. Black curves correspond to a linear waveguide (or weak-intensity SPP modes propagating in a nonlinear waveguide). It is evident that the presence of self-focusing nonlinearity profoundly alters the dispersion of SPPs, giving birth to backward-propagating modes and critical points in the SPP density of states. The backward-propagating modes exhibit a negative group velocity and are described by portions of the dispersion curves with negative slopes. As shown by a thin horizontal line, a nonlinear medium as an example, the dispersion relations reveal in the case of a self-focusing nonlinearity the existence of backward-propagating modes as well as the critical points in their energy spectra where the density of states is enhanced. In the case of a self-defocusing nonlinearity the existence of backward-propagating modes at certain frequencies, but it gives rise neither to backward-propagating modes nor to the critical points (the density of states corresponding to the needle-shaped sections of the dispersion curves is finite). Another fundamental difference of the curves in Fig. 3 from those in Fig. 2 is that they exhibit a cutoff in the reciprocal space for large $\beta$. The cutoff frequency and wave number strongly depend on $E_0$ and $h$, as do the positions of critical points. Similar to the case of $\alpha > 0$, the modes of a specific symmetry that coexist at the same frequency differ by the electromagnetic field intensities associated with them. It may also interest the reader to note the difference between SPP dispersion in a metallic slab surrounded by a dielectric, and that in a gap plasmonic waveguide. In the first instance, the nonlinear dispersion relations only marginally differ from their linear analogs, whereas in our case a drastic reconstruction of the SPP energy spectrum occurs even for small nonlinear coefficients, because of the different heterostructure topology.

In summary, SPP dispersion relations have been derived for plasmonic waveguides with an arbitrary power-law nonlinearity within the dielectric core. When applied to a Kerr medium as an example, the dispersion relations reveal in the case of a self-focusing nonlinearity the existence of backward-propagating modes as well as the critical points in their energy spectra where the density of states is enhanced. In the case of a self-defocusing nonlinearity, backward-propagating modes do not exist. Moreover, the density of states is not enhanced at the critical points, which now indicate a power-dependent cutoff of the forward-propagating modes. We note in passing that Eq. (4) can also be used to study symmetry-breaking bifurcations of SPP modes, if the assumption of a particular mode symmetry in Eq. (2) is discarded. This topic requires further investigation.

This work was supported by the Australian Research Council, through its Discovery Grant scheme under Grant No. DP110100713. The work of G.P.A. was also supported by the US National Science Foundation (ECCS-0801772).

---

\[^{1}\text{J. A. Schuller, E. S. Barnard, W. Cai, Y. C. Jun, J. S. White, and M. L. Brongersma, Nat. Mater. 9, 193 (2010).}\]


\[^{3}\text{S. A. Maier, Plasmonics: Fundamentals and Applications (Springer, Berlin, 2007).}\]

\[^{4}\text{J. B. Pendry, D. Schurig, and D. R. Smith, Science 312, 1780 (2006).}\]

\[^{5}\text{D. A. B. Miller, IEEE Proc. 97, 1166 (2009).}\]