Indistinguishability of orbital angular-momentum modes in spontaneous parametric down-conversion

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(Received 3 December 2008; revised manuscript received 26 January 2009; published 8 May 2009)

Identification of light modes usually requires careful considerations of the collecting geometry. This is particularly true for states carrying orbital-angular-momentum modes from spontaneous parametric down-conversion due to the entanglement of the signal and idler fields. A detailed understanding of the generated modes in a general case allows the design of efficient detection setups. This is true for distinct cases, e.g., when multiple samplings are performed by repeated state generation or in single-photon cryptography or quantum computation, where a photon state is generated within a short-time window and a single measurement is performed. Aspects of nonorthogonality of signal and idler modes and effects of restrictions on the collecting geometry are discussed in this work.

DOI: 10.1103/PhysRevA.79.055805  
PACS number(s): 42.50.Tx, 42.50.Dv, 42.50.Ex

States of light carrying orbital angular momentum (OAM) are defined within spatial regions and their measurements require a careful probing of these regions to allow state identification. Sometimes, space limitations are not detrimental for state identification; in others, they are crucial. Light spin or circularly polarized light states, for example, needs a single wave-vector approximation because “spin” is associated with a specific wave vector. Distinctly, OAM states demand finite regions of space for identification: a bundle of wave vectors is needed to specify a given mode. While this is an undisputed statement, sometimes specification of this geometry is not a clear task and even the way one defines the photon concept plays a role. Figure 1 depicts wave vectors associated to an orbital angular-momentum mode near to a focal plane. Association of wave vectors with an orbital angular-momentum spatial mode \( U_i(r,t) \) can be seen from the quantized electric field \( \hat{E}^{(+)}(r,t) \) decomposed in the plane-wave basis,

\[
\hat{E}^{(+)}(r,t) = \sum_k l^*_k(k') \mathbf{e}_k \hat{a}_k e^{i(k \cdot r - \omega_k t)},
\]

where \( \mathbf{e}_k \) is the unitary polarization vector and \( l^*_k(k') = -i \sqrt{\hbar / 2eV} \). Decompose \( U_i(r,t) \) in the set of unitary matrices \( U_{k,l} \) transforming between the two representations,

\[
U_i(r,t) = \sum_k U_{k,l} e^{i(k \cdot r - \omega_k t)},
\]

they obey \( \sum_k U_{k,l} U_{k,l}^* = \delta_{kk'} \). Writing \( l^*_k(k') \mathbf{e}_k \hat{a}_k = \sum_k \delta_{kk'} l^*_k(k') \mathbf{e}_k \hat{a}_k \) and replacing it in \( \hat{E}^{(+)}(r,t) \), one obtains

\[
\hat{E}^{(+)}(r,t) = \sum_l \sum_{k'} U^*_{l,k'} l^*_l(k') \mathbf{e}_k \hat{a}_k, \quad \left[ \sum_l U_{k,l} e^{i(k \cdot r - \omega_k t)} \right]
\]

\[
= \sum_l \hat{c}_l U_l(r,t).
\]

In order to guarantee that an annihilated photon belongs to the spatial mode \( U_i(r,t) \), the collection setup has to accept the set of wave vectors defining \( \hat{c}_l = \sum_k U_{k,l} e^{i(k \cdot r - \omega_k t)} \) [or in \( U_l(r,t) = \sum_k U_{k,l} e^{i(k \cdot r - \omega_k t)} \)]. In cases where the experimental setup constrains the collected photon to a smaller set of wave vectors, what would be the effect on the distinguishability of different OAM modes in spontaneous parametric down-conversion (SPDC)? Understanding the generated modes in SPDC is necessary to better understanding this problem as well as the SPDC phenomenon itself. A brief presentation of SPDC will be done to introduce the necessary definitions and equations.

The phenomenon of SPDC has been well studied for some decades; it is described by the wave state [1]

![FIG. 1. (Color online) Wave vectors k and k’ around the focus in a mode with angular momentum l ≠ 0; they do not cross the z axis. Collecting a subset of the wave vectors defining the mode may lead to a poor distinguishability between modes with different l (e.g., detecting within area A_coll).](image-url)
\[ |\psi(t)\rangle = |0\rangle + \sum_{\alpha,\beta} \int d^3 k' \int d^3 k A_{k,\beta} (\alpha,\gamma) f^{(\gamma)}_E (\alpha k) \times d^{(\alpha)}_E (\alpha k) T(\Delta \omega) \tilde{\psi}_P (\Delta k) \tilde{a} (\kappa, \alpha) \tilde{a} (\kappa', \alpha') |0\rangle, \]

(4)

where two down-converted photons are generated in an entangled state, \( A_{k,\beta} (\alpha,\gamma) \), which is the amplitude of the nonlinear polarizability that depends on the unitary polarization vectors (\( \alpha \) polarized) as well as on the nonlinear susceptibility, \( T(\Delta \omega) = \text{exp}(i \Delta \omega (t - t_{in}/2)) \text{sin}(\Delta \omega t_{in}/2) / (\Delta \omega / 2) \) is the time window function defining the \( \Delta \omega \) range given the interaction time \( t_{in} \), \( \Delta \omega = \omega_k + \omega_p - \omega_p, \Delta k = k + \tilde{k} - k_p \). For \( t_{in} \rightarrow \infty, T(\Delta \omega) \rightarrow \pi \delta(\Delta \omega) \). \( \tilde{\psi}_P (\Delta k) = \int V d \tau \psi_P (\tau) \exp(-i \Delta k \cdot r) \) and \( \psi_P \) is the field amplitude in \( E(\rho, \phi, z, t) = \psi_P (\rho) e^{i(k_x x - \omega_p t)} \). For a pump beam with orbital angular momentum \( l \) \cite{1}, one has

\[ \psi_P (\rho, \phi, z) = \frac{A_l \rho}{1 + (z / z_R)^2} \left[ \frac{\rho^2}{w(z)^2} \right] \left[ z - \frac{\rho^2}{2 (z^2 + z_R^2)} + l \arctan(y/x) \right] \times \exp \left[ - \frac{\rho^2}{w(z)^2} \right] \times \exp \left[ i (2l + 1 + l \arctan(z / z_R)), \right] \]

(5)

\( L_i^* \) is the associated Laguerre polynomial and \( \rho^2 = x^2 + y^2 \).\( w(z) \) is the beam waist in a generic position \( z; z_R \) is the Rayleigh range. The spectral function \( \tilde{\psi}_P (\Delta k) \) carries the specific pump mode that excites the nonlinear crystal. Using cylindrical coordinates \( (\rho, \phi, z) \), where \( d\tau = \rho d\rho d\phi dz \), the integrals in \( \psi_P (\Delta k) \) are performed giving \cite{2}

\[ \tilde{\psi}_P (\Delta k) = \pi A_l \rho (i/2)^l \left( z_R / k_p \right)^{l/2} e^{-il(l+1/2)} \left[ \frac{k^2}{z_R} \right] \left[ \frac{k^2}{z_R} \right] \times e^{-z_R^2 / 2k_p^2 \rho^2} \left[ \frac{\rho}{w(z)^2} \right] \left[ \frac{\rho^2}{w(z)^2} \right] \times \exp \left[ - \frac{\rho^2}{w(z)^2} \right] \times \exp \left[ i (2l + 1 + l \arctan(z / z_R)), \right] \times e^{-i \Delta k \cdot z} e^{-i \rho^2 / 2l \arctan(z / z_R)} e^{i(1+2l) \arctan(z / z_R)} \right] \right] \times \int d^3 k \int d^3 k' e^{i(\Delta k \cdot z) / \left( 1 + z_R^2 \right)} \int d^3 k \int d^3 k' \left| A_{\Delta k'}^\text{amp} \right|^2 \left| 1_{k'} \right| \left| 1_{k} \right| \]

(6)

where \( l_c \) is the crystal thickness. With the usual condition \( z_R \gg l_c \), one obtains

\[ \tilde{\psi}_P (\Delta k) = \pi A_l \rho (-1) i (i/2)^l \left( z_R / k_p \right)^{l/2} e^{-il(l+1/2)} \times e^{i(1+2l) \arctan(z / z_R)} e^{-i \rho^2 / 2l \arctan(z / z_R)} \times \frac{\sin \left( \Delta k \cdot z / 2 \right)}{\Delta k / 2} \left| L_i^* (\xi) \right|, \]

(7)

where \( \xi = (z / k_p) \rho^2 \) and \( p_\lambda = \sqrt{\Delta k^2 + \Delta k^2}. \)

It should be observed that in the wave-vector space, one could identify \( \Delta k / z_R \) with an angle \( \phi_2 \), associated with the signal and idler pair. Differently from the starting angle \( \phi = \arctan(y/x) \) associated with the pump mode and \( r \)-space variables, \( \phi_2 \) connects \( k \)-space variables from signal and idler. In some simple cases (e.g., type I, degenerate case) \( \phi_2 = (\phi + \phi') / 2 \), where \( \phi \) and \( \phi' \) are individual azimuthal angles for signal and idler in the wave-vector space, respectively, around the pump propagation direction—taken as the quantization axis.

This is a convenient point to discuss nonorthogonality of the signal and idler fields. Laguerre modes are orthogonal satisfying the relation \( \int \rho L_{\Delta k}^\text{amp} (\rho) L_{\Delta k'}^\text{amp} (\rho) e^{i \rho \Delta \omega / 2} d\rho = 0 \) for \( m \neq n \); one could ask, e.g., if \( L_{\Delta k}^\text{amp} (\xi) \) can be decomposed into products of Laguerre modes describing signal and idler as independent contributions. The negative answer can be physically understood due to the signal and idler entanglement. From the definitions of \( \Delta k \) and \( \xi \), it can be seen that expansion of \( L_{\Delta k}^\text{amp} (\xi) \) into products of independent terms in \( k \) and \( k' \) cannot be done. Further processing or mode projections (filtering) of the SPDC generated light have to be applied to extract signal or idler in pure Laguerre modes \cite{4}.

\[ \left| \psi_{\Delta k} (\Delta k) \right|^2 \text{is directly proportional to the crystal probability to generate signal and idler pairs with wave vectors } k \text{ and } k' \text{ given the paraxial incident pump field mode at } k_p \text{ (as given by Eq. (5)). In this theory, differently from usual treatments, the outgoing signal and idler fields are not limited to paraxial cases. It is understood that at each successful down-conversion event, the excited crystal decays onto a single-photon pair, along any emission directions } k \text{ and } k' \text{ allowed by conservation laws. These laws define the so-called phase matching conditions that specify maxima for Eq. (7). Basically, the maximum of } \text{sin}(\Delta k / 2) / (l / \Delta k / 2) \text{ is given by values of the longitudinal variables } \Delta k = (k + k' - k_p), \text{ while maximum of } e^{-\xi^2 / 2} L_i^* (\xi) \text{ is given by values of the transverse variables connected to } \xi = (z / k_p) (\Delta k^2 + \Delta k^2). \text{ Both } \Delta k \text{ and } \xi \text{ will depend on the azimuthal angles if the refractive index has azimuthal dependence [3]. Whenever this dependence exists, the transfer of OAM from the pump beam to the down-converted photons may be frustrated as shown in [3,5]. The situation described in this work is not related to frustrated OAM transfer.}

A normalized wave state describing any entangled photon pair from SPDC can be written \cite{3}

\[ \tilde{\psi}_P (\xi)_{\text{in}} = \int_k d^3 k \int d^3 k' \frac{A_{\Delta k}^{\text{amp}}}{\sqrt{\int_k d^3 k' |A_{\Delta k'}^{\text{amp}}|^2}} \left| 1_k \right| \left| 1_{k'} \right|, \]

(8)

\( A_{\Delta k}^{\text{amp}} = \text{exp} \left( -i (i / 2)^2 |L_{\Delta k}^\text{amp} (\xi)| e^{i\Delta k \cdot (k' - k)} e^{-il(l+1/2)} \right) \)

will be specified by phase matching according to \( k' = k_p + \Delta k - k \) where \( \Delta k \) is taken at the value that maximizes \( A_{\Delta k}^{\text{amp}} \). OAM indexes for signal and idler photons are not present in \( |1_k \rangle \) (say, \( l_s \)) or \( |1_{k'} \rangle \) (say, \( l_i \)). An OAM mode with index \( l \) has a mode signature distinct from a single-plane-wave signature. In Eq. (8) the effect of the nonlinear polarizability \( A_{k_r} (\alpha, \beta) \) has been neglected; it provides only smooth contributions along the down-converted rings.

In optical systems used to transmit or collect SPDC states, cases range from free-propagation geometry, where in a distance of, say \( D = 10 \text{ km}, \) the wave-front dimension for a coincidence donutlike structure will spread by \( D \tan 0.01 \approx 100 \text{ m} \) and telescopic guidance is necessary
FIG. 2. (Color online) Used fraction of the scattered light as a function of $\xi_{\text{cut}}$ for $l=0,1,2,3,4$.

(e.g., [6] with a simple $l=0$ case), Dove prisms [7] to select OAM modes ($l \neq 0$), and cases of small optical structures where microguidance is necessary [8] ($l \neq 0$).

In spherical coordinates $k = k(k, \varphi, \theta)$; imposing geometrical restrictions $\Delta k$, $\Delta \varphi$, and $\Delta \theta$ on the detected light field may lead to a poor mode distinguishability. Rough estimates can be done without specifying particular cases and involves direct considerations over $\xi$ and $\Delta k$. However, it is hard to use this approach to map it onto specific cases. The variables $\xi$ and $\Delta k$ are related to structures in the signal and idler wave-vector space [3]. $\xi$ connects variables $(k, k', \varphi, \varphi')$ and constrains them by phase match in the nonlinear medium (see Ref. [3]). A detecting system that restricts too much the collection area (e.g., see area $A_{\text{rc}}$ in Fig. 1) is not able to reveal the orbital angular-momentum value $l$ associated to the incoming mode. An estimate of restrictions on the range of wave vectors collected is sketched in Fig. 2 for $l$ values ranging from $l=0$ to $l=4$. This plot represents the magnitude square of the amplitude in Eq. (8) integrated from 0 to an arbitrary value of $\xi = \xi_{\text{cut}}$. This gives a good idea of the effect of a wave-vector cutoff. However, it should be emphasized that the calculation for specific geometries needs specific wave-vector boundaries and cannot be done with the compact $\xi$ coordinate. A complementary view is shown in Fig. 3 where the abscissa axis gives the $l$ values for the same range of $\xi_{\text{cut}}$ values.

While Fig. 1 shows a general collecting area, particular shapes are involved in an experiment. Restriction in the collecting geometries may be exemplified by stops with azimuthal symmetry (see Fig. 4) or sector shapes that break this symmetry (see Fig. 5).

Even using a correct OAM filter (e.g., a proper OAM mask) but with a $k$-restricting collection optics, some degree of indistinguishability appears. For simplicity, assume that one of the detecting systems (either signal or idler) detects the correct mode while the other has a restrictive collection optics as depicted in Figs. 4 and 5. Consider that a perfectly absorbing stop is being used in front of a detection system blocking a range of wave vectors from the generated SPDC.

In a simplified way, the accepted fraction can be written

$$|\psi_{\text{coll}}(\xi)|^2 = \int \int \int d^3k \sqrt{d^3k'} A_{\text{amp}}^{(l,p)} |1_k\rangle \times |1_k\rangle,$$

where the symbol $\int$ indicates a partial integration representing wave vectors allowed by the stopper (this simplification just tries to avoid OAM decomposition to describe the wave vectors reaching the detector). A measure of the overlap between these states is

$$F(\rho_{\text{coll}}, \rho_{\varphi}) = \left| \langle \rho_{\text{coll}} | \rho_{\varphi} \rangle \right| = \frac{\int \int d^3k \int d^3k' |A_{\text{amp}}^{(l,p)}|^2}{\int d^3k \int d^3k' |A_{\text{amp}}^{(l,p)}|^2}.$$

The fraction represented by Eq. (10) represents the best possible identification of the generated wave state due to partial

FIG. 3. (Color online) Used fraction of the scattered light as a function of $l$ for some values of $\xi_{\text{cut}}$ up to $\xi_{\text{cut}}=30$.
FIG. 6. (Color online) A $\chi^{(2)}$ crystal is pumped by a uv laser mode at $\lambda_p=3511$ Å, with OAM $l=2$. Type I signal and idler rings and the calculated coincidence structure at the idler position (wave-vector space) given that a signal photon has been detected at $k$. The coincidence structure corresponds to OAM $l=2$. The ring angles are at $\theta_0=0.49$ rad and $\theta'_0=0.54$ rad.

detection imposed by the detecting geometries. As a simple example, assume that a nonlinear crystal has been excited by a laser mode carrying OAM $l$ and a down-converted signal photon has been detected at a particular wave vector $k$. It is also assumed that this interaction has azimuthal symmetry by considering type I SPDC [5]. The assumption of a specific $k$ vector (a plane wave) implies that this photon has $l=0$ and, therefore, the idler photon has to carry the OAM $l=1$ [5].

However, the detecting system for the idler photon has to allow detection of a range of wave vectors that, in principle, could reveal that the single idler photon could belong to an OAM mode. Restrictions on the wave vectors collected will be imposed by the use of stoppers either with azimuthal symmetry (see Fig. 4) or without azimuthal symmetry (see Fig. 5). As an example, a type I crystal potassium titania oxide phosphate (KTP) is chosen with ordinary refractive indexes for signal and idler photons, pump-laser wavelength $\lambda_p=3511$ Å with OAM $l=2$, signal $\lambda_s=6328$ Å and idler $\lambda_i=6870$ Å; and $L=0.2$ cm and $z_p=10$ cm. Angles internal to the medium will be considered for simplicity (Snell’s law easily give external angles). Polar angles $\theta$ and $\theta'$ for signal and idler that maximize the function $A_{\text{amp}}^{(l,p)}$ can be found numerically. They give the position of the signal and idler rings at their maximum amplitudes. Choosing $\varphi=0$ around the signal ring as the wave-vector position where a signal photon is found and looking around the $\varphi'=\pi$ on the idler ring, the coincidence structure given by $|A_{\text{amp}}^{(l,p)}|$ is calculated. Figure 6 shows the signal and idler rings and the calculated coincidence structure at the idler position. Figure 7 shows $F$ for a stopper allowing light within annular rings centered around the idler ring as a function of spreads $\Delta\theta$ and for a stopper allowing light in sectors with openings $\Delta\varphi$. Figure 7 summarizes the obtained results, where the overlap $F$ represents the maximum allowable information that can be obtained due to geometrical restrictions. Differently from identifying polarization states, OAM states demand that a set of wave vectors could be detected to allow reasonable identification of the precise OAM value attached to the mode. This is the case when repeated mode preparations are possible or even when a single sampling is allowed [9]. In Ref. [9], no detail on geometrical restrictions can be extracted and were—in fact—of no interest, differently from the results of the present Brief Report. A study of modifications introduced by the use of spatial light modulators on the detection path of SPDC can be seen in [10] aimed to the study of Fourier relationship between angle and angular momentum. These results call the attention to the fact that when designing a collecting system for OAM modes, a careful consideration of the range of wave vectors has to be taken into account. Application for any other $l$ value is straightforward.

This work was supported by the U. S. ARO MURI under Grant No. W911NF-05-1-0197 on Quantum Imaging.