Noise initiation of stimulated Brillouin scattering

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We describe a theoretical model that shows how stimulated Brillouin scattering (SBS) is initiated by thermally excited acoustic waves distributed within a Brillouin-active medium. This model predicts how the SBS reflectivity, Stokes linewidth, and fluctuations in Stokes intensity depend upon the laser intensity and upon the physical properties of the SBS medium. This model also leads to the prediction that the value of the single-pass gain (i.e., $G = gIL$) at the threshold for SBS is not a universal number, but depends upon the laser frequency and on the properties of the SBS medium.

For typical organic liquids at room temperature, $G$ is in the range 20–25.

I. INTRODUCTION

Stimulated Brillouin scattering (SBS) is a process in which a laser and Stokes wave interact with one another in a material medium (see Fig. 1) through the intermediary of a sound wave, leading to the amplification of the Stokes wave and the attenuation of the laser wave. The SBS process can be treated theoretically in a very straightforward manner for the case of an SBS amplifier, since the laser and Stokes fields are both applied externally. The theoretical analysis is much more complicated for the case of an SBS generator, because only the laser field is applied externally and the Stokes wave is created within the medium. In this case, the process of SBS is “seeded” by spontaneous Brillouin scattering from thermally excited sound waves or (at low temperatures) by quantum noise. It is known empirically that there is a threshold for the occurrence of SBS generation, and that the threshold condition is that the single-pass gain through the Brillouin medium, $G = gL_0L$ [here $g$ is the SBS gain factor that is defined in Eq. (22) below, $L_0$ is the intensity of the incident laser field, and $L$ is the length of the Brillouin medium], must exceed some threshold value $G_{th}$, which is typically of the order of 25.

It is the intent of the present paper to examine in detail how the SBS process is initiated from noise. In particular, we develop a model which shows how SBS is initiated by a spatially distributed, fluctuating noise source. We compare the predictions of this model with those of a simpler model due to Zel’dovich, Plipetskii, and Shkunov (which we will henceforth call the localized, nonfluctuating source model) that ascribes the origin of SBS to spontaneous scattering occurring in a thin region located near the rear of the SBS medium (near $z = L$ in the notation of Fig. 1). The predictions of our new model (which we will henceforth call the distributed, fluctuating source model) are in good qualitative agreement with those of the localized, nonfluctuating source model, which themselves are in good qualitative agreement with experimental observations. However, we find that the two models lead to measurably different predictions for certain properties, such as the dependence of the SBS reflectivity on the single-pass gain through the interaction region. These differences result both from the distributed nature of the initiation and as a consequence of fluctuations in the initiation process. In addition, our model allows us to treat noise properties of the SBS process, such as the linewidth and intensity fluctuations in the output intensity.

II. THE EQUATIONS DESCRIBING SBS

In this paper, we consider SBS in the backward direction in the geometry shown in Fig. 1. We assume that the total optical field within the Brillouin-active medium,

$$\tilde{E}_{tot}(z,t) = \tilde{E}_L(z,t) + \tilde{E}_S(z,t),$$

(1)

can be represented as the sum of a laser component

$$\tilde{E}_L(z,t) = \frac{1}{2}E_L(z,t)e^{(ik_Lz - \omega_Lt)} + \text{c.c.}$$

(2)

propagating in the positive z direction, and a Stokes component

$$\tilde{E}_S(z,t) = \frac{1}{2}E_S(z,t)e^{(ik_Sz - \omega_St)} + \text{c.c.}$$

(3)

propagating in the negative z direction. Note that we are
III. FORM OF THE LANGEVIN NOISE TERM

We next examine the statistical properties of the Langevin noise term appearing in Eq. (9). To do so, it is convenient to turn to a discrete description of the acoustic field. We divide the interaction region into subregions of length $\Delta z$, and let $\rho_i$ denote the amplitude of the acoustic disturbance averaged over one such subregion. In the absence of the electrostrictive driving term, this quantity obeys the equation

$$\frac{d\rho_i}{dt} + \frac{1}{2} \Gamma \rho_i = f_i(t),$$

where $f_i(t)$ represents the quantity $f(z,t)$ averaged over subregion $i$. We make the standard physical assumptions that $f_i$ is a Gaussian random variable with zero mean such that $\langle f_i \rangle = 0$ and is $\delta$ correlated in the sense that

$$\langle f_i(t)f_j^*(t') \rangle = Q \delta_{ij} \delta(t - t').$$

We next determine the value of the parameter $Q$ which characterizes the strength of the fluctuations in $f_i$. By introducing the formal solution of Eq. (10) into the left-hand side of Eq. (11), we find that

$$\Gamma \langle \rho_i(t)\rho_i^*(t) \rangle = Q \delta_{ij}.$$  

We next calculate the density correlation $\langle \rho_i(t)\rho_i^*(t) \rangle$ by considering the energy stored in a thermally excited acoustic wave. The energy density of a sound wave is given by

$$\langle u \rangle = \frac{1}{2} \rho_0 \langle V^2 \rangle + \frac{1}{2} \rho_0 \langle \bar{\rho}^2 \rangle / \rho_0,$$

and hence the energy in slab $i$ whose volume is $A\Delta z$ is equal to $\langle u \rangle A \Delta z$ or to

$$E_i = \langle \frac{1}{2} \rho_0 \langle V^2 \rangle + \frac{1}{2} \rho_0 \langle \bar{\rho}^2 \rangle / \rho_0 \rangle A \Delta z.$$  

According to the equipartition theorem, each term in Eq. (14) contributes an amount $\frac{1}{2}kT$ to the energy. We hence find that $\langle \bar{\rho}^2 \rangle = kT \rho_0/v^2 A \Delta z$. However, since $\langle \bar{\rho}^2 \rangle = \frac{1}{2} \rho_i(t)\rho_i^*(t)$, we find that

$$\langle \rho_i(t)\rho_i^*(t) \rangle = 2kT \rho_0 / v^2 A \Delta z.$$  

By comparison with Eq. (12), we see that the strength parameter is given by

$$Q = \frac{2kT \rho_0 \Gamma}{v^2 A \Delta z}.$$  

If we now return to a continuum description of the acoustic field, we find that the Langevin source term obeys the relations

$$\langle f(z,t) \rangle = 0$$

and

$$\langle f(z,t)f^*(z',t') \rangle = Q \delta(z - z') \delta(t - t'),$$

where

$$Q = \frac{2kT \rho_0 \Gamma}{v^2 A \Delta z}. $$
We can convert this last formula into its quantum-mechanical analog by replacing $kT$ by $\hbar \Omega(\bar{\eta} + 1)$, where $
bar = (e^{kT/\hbar \Omega} - 1)^{-1}$ gives the mean number of phonons per mode of the acoustic field. We thereby find that

$$Q = \frac{2\rho_0 \Gamma \bar{\eta} |1 + \bar{\eta}|}{v^2 A}.$$  

(19)

This expression for $Q$ along with Eqs. (5), (9), and (17) completely describes the SBS process including its initiation by noise.

$$E_S(z, \tau) = E_S(L, \tau) + \left[ \frac{\Gamma G(L - z)}{4 L} \right] \frac{1}{2} \int_0^\tau d\tau' e^{-\Gamma(\tau - \tau')/2} E_S(L, \tau') I_1 \left[ \frac{1}{2} \left| \frac{\Gamma G(L - z)}{(\tau - \tau')/L} \right| \right] / (\tau - \tau')^{1/2}$$

$$+ \frac{\gamma \omega_S}{4 \rho_0 n c} \int_0^\tau d\tau' \int_L^L d\tau'' e^{-\Gamma(\tau - \tau'')/2} f(z', \tau') G(z - z') \left[ \frac{1}{2} \left| \frac{\Gamma G(z - z')}{(\tau - \tau'')/L} \right| \right] ,$$

(20)

where $I_j(x)$ is the modified Bessel function of order $j$, $r = t - zn/c$ is the local time variable, and where we have introduced the quantity

$$G = \frac{\gamma^2 n_0^3}{4\pi n c^2} E_L^2 L,$$

(21)

which we shall see below is the logarithm of single-pass intensity amplification. It is also convenient to introduce the SBS gain factor $g$, defined through the relation $G = gIL$, where $I = (nc/8\pi)|E_L|^{2}$ is the laser intensity. We hence find that the gain factor is given by the expression

$$g = \frac{\gamma^2 \omega_S^2}{\rho_0 n c^2 \Gamma}.$$

(22)

In the absence of an input Stokes field, only the laser term in Eq. (20) gives a nonvanishing contribution to the expectation value of the Stokes intensity. (The same conclusion holds quantum mechanically for the normally ordered intensity when the Stokes input is in the vacuum state.) We then find, letting $\tau \to \infty$, and making explicit use of the statistical properties of the Langevin noise term as given by Eqs. (17) and (19), that the Stokes intensity is given by

$$\langle |E_S(0)|^2 \rangle = \left[ \frac{\gamma \omega_S}{4 \rho_0 n c} \right]^2 |E_L|^2 Q \int_0^\infty d\tau' \int_L^L dz' e^{-\Gamma/2} I_0^2 \left[ \frac{1}{2} \left| \frac{\Gamma G(z - z')}{L} \right| \right].$$

(23)

The integrations can be performed explicitly, leading to the result

$$\langle |E_S(0)|^2 \rangle = \left[ \frac{\gamma \omega_S}{4 \rho_0 n c} \right]^2 |E_L|^2 Q \frac{OL}{\Gamma} \times e^{G/2} [I_0(G/2) - I_1(G/2)].$$

(24)

Through use of Eqs. (18), (22) and (24), we now see that the SBS reflectivity defined by $R = \langle |E_S(0)|^2 \rangle / |E_L(0)|^2$ is given by

$$R = \frac{Y e^{G/2}}{\sqrt{\pi}} [I_0(G/2) - I_1(G/2)],$$

(25)

where we have introduced the parameters

$$Y = (\bar{\eta} + 1)X, \quad X = g \hbar \omega_S \Gamma \frac{L}{4A}.$$

(26)

Equations (25) and (26) give the formal solution to the equations describing the initiation of SBS by a distributed noise source. This solution is displayed graphically in Fig. 2 as the curve labeled distributed fluctuating source. For comparison, the growth rate $\exp(G)$ that would be experienced by a localized source near $z = L$ is shown as the curve labeled localized source. In plotting this figure, we have chosen the amplitude of the localized source such that the two curves agree in the limit of low gain. In this limit ($G \ll 1$), the SBS reflectivity predicted by Eq. (25) becomes simply $R = Y$. In this case the reflected field originates entirely from the scattering of the laser field by spontaneously (i.e., thermally) generated phonons, and hence the reflectivity is independent of the laser intensity. In the opposite limit of high gain ($G \gg 1$) the reflectivity becomes

$$R = \frac{Y}{\sqrt{\pi}} e^{G/2}.$$  

(27)

This result shows that, due to the distributed nature of the noise source that initiates SBS, the effective amplification of the SBS generation process is reduced by...
most 1. Our theory hence predicts that at room temperature the threshold for SBS in organic liquids corresponds to \( G \) in the range 20--25.

The theoretical prediction for the numerical value of the gain at threshold is very different for the case of SBS in a single-mode optical fiber. In such a case, the parameter \( L \) appearing in Eq. (6) represents the physical length of the fiber and \( A \) represents the physical area of the region of the fiber in which the radiation is confined. Consequently the factor \( L/4A \) can be much larger than the value \( 1/\lambda \) which applies to free propagation through a region of unit Fresnel number. For this reason the gain at threshold can be much lower for an optical fiber than for the case of an organic liquid. For one particular case, Ip- pen and Stolen\(^{14}\) report a value of \( G \) at threshold of 20.

V. LOCALIZED NONFLUCTUATING SOURCE MODEL OF THE INITIATION OF SBS

In this section we summarize briefly the localized, nonfluctuating source model of Zel'dovich, Piplietskii, and Shkunov\(^{10}\) for the initiation of SBS. In subsequent sections, the predictions of this model will be compared with those of the present calculation. The model of Zel'dovich, Piplietskii and Shkunov considers SBS under steady-state conditions; in such a case it is adequate to describe SBS in terms of coupled intensity equations for the laser and Stokes fields, which are given by

\[
\frac{dI_L}{dz} = -gI_L I_S, \quad \text{(28a)}
\]
\[
\frac{dI_S}{dz} = -gI_L I_S, \quad \text{(28b)}
\]

where \( g \) is the SBS gain factor given by Eq. (22). We solve these equations subject to the boundary conditions that \( I_L(0) \) is specified and that \( I_S(L) = f I_L(L) \). The boundary condition at \( z = L \) is a statement that SBS is initiated by spontaneous Brillouin scattering occurring in a small region near the rear of the interaction volume; \( f \) then represents the fraction of the transmitted laser intensity \( I_L(L) \) that is backscattered to form the Stokes seed. Under typical conditions involving SBS in organic liquids, \( f \) is of the order of \( 10^{-12} \). To good approximation (i.e., ignoring terms of order \( f \)) the solution to the coupled intensity equations (28) for these boundary conditions is

\[
G = \frac{\ln R + G_{ref}}{1 - R}, \quad \text{(29)}
\]

where, as before, \( G = gI_L(0)L \) is the single-pass gain, \( R = I_S(0)/I_L(0) \) is the SBS reflectivity, and where \( G_{ref} = -\ln f \). This functional form is plotted in Fig. 3(a) for the typical case of \( G_{ref} = 28.5 \) as the curve labeled localized, nonfluctuating source. We see that the reflectivity \( R \) is much less than unity for \( G < G_{ref} \) and increases rapidly with \( G \) for \( G > G_{ref} \). For this reason, \( G_{ref} \) is a measure of the threshold gain required for the occurrence of SBS.
VI. NUMERICAL INTEGRATION OF THE SBS EQUATIONS

In order to study the properties of SBS in the general case where the SBS reflectivity is not necessarily much smaller than unity, we have performed a numerical integration of the complete set of coupled equations describing the laser, Stokes, and phonon fields [Eqs. (5) and (9)] with the Langevin source term given by Eqs. (17) and (19). We divide the interactions region into 800 subregions and integrate for a length of time sufficient to ensure that transients associated with the turn on of the laser field have died away. Unlike the case of the constant-pump theory, where universal results such as those shown in Fig. 2 could be obtained, in the present case the predictions depend upon the values of the initiation parameter $Y = (\hat{n} + 1)X$ and on the product $\Gamma T_i$ of the phonon damping rate $\Gamma$ and the transit time $T_i = nL/c$ through the medium. The results of such calculations have been used to obtain Figs. 3–6, which are described in detail below.

VII. COMPARISON OF VARIOUS MODELS FOR THE INITIATION OF SBS

Figure 3 shows the dependence of the SBS reflectivity $R$ on the single-pass gain $G$ according to various models for the initiation of SBS. In all cases we consider SBS in carbon disulfide at room temperature, in which case the initiation parameter $Y$ defined by Eq. (26) is equal to $1.08 \times 10^{-10}$. The curves labeled distributed, fluctuating source (numerical integration) give the results obtained by

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FIG. 3. SBS reflectivity $R$ plotted as a function of the single-pass gain $G$, according to the theoretical models described in the text.

FIG. 4. Fluctuations in the Stokes output intensity, as predicted by the numerical integration of the equations of the distributed fluctuating source model described in the text. As the single-pass gain $G = gIL_0L$ is increased, the mean reflectivity increases and the fluctuations tend to become suppressed due to pump depletion effects. The case $\Gamma T_i = 20$ is shown.
results given in the curve labeled localized, fluctuating source. This curve was obtained by averaging the reflectivity predicted by Eq. (29) over the input intensity distribution of a thermal source whose mean intensity is the same as that used in obtaining the curve labeled localized, nonfluctuating source. This calculation was performed using the method outlined in the Appendix. We see that the primary effect of including source fluctuations in the localized source model is to shift the curve slightly to the right. We can shift the curve back to the left by using a smaller value of \( G_{\text{ref}} \) in the theoretical model. This dependence on \( G_{\text{ref}} \) is illustrated in part (b) of the figure. We see from part (a) that the localized nonfluctuating source model with \( G_{\text{ref}} = 28.5 \) agrees well with the results of the full numerical calculation near threshold, and we see from part (b) that the localized fluctuating source model with \( G_{\text{ref}} = 25.5 \) also agrees well near threshold; however, in neither case can we obtain good agreement for all values of \( G \).

VIII. FLUCTUATIONS IN THE STOKES OUTPUT INTENSITY

Since the SBS process is initiated by noise, the Stokes output intensity is expected to be a fluctuating quantity. Figure 4 shows the time evolution of the output intensity as predicted by our numerical integration of the SBS equations. The case \( \Gamma T_r = 20 \) is treated. We see that the fluctuations in the output intensity tend to be suppressed through the use of a large value of the single-pass gain \( G \). This behavior occurs as a result of the saturable nature of the SBS gain due to pump depletion effects. We also see that the time scale of the fluctuations increases as \( G \) is increased, due to spectral gain narrowing of the Stokes radiation. In the limit of high gain, the fluctuations that do occur have a characteristic, bidirectional, spiked shape that is reminiscent of the phase waves that have been predicted to occur in stimulated Raman scattering.\(^{15}\) We have determined that these features are associated with an abrupt change in the phase of the Stokes wave by approximately \( \pi \) radians. The nature of these features is described further by Gaeta.\(^{16}\) Our results are similar to those of Wandzura\(^{17}\) and to Dianov et al.\(^{9}\)

We characterize the strength of the Stokes intensity fluctuations by calculating the variance of the Stokes intensity, which is defined by

\[
\text{var}(I_S) = \langle I_S^2 \rangle - \langle I_S \rangle^2.
\]

In Fig. 5 we show the dependence of the square root of the normalized variance, \([\text{var}(I_S)/\langle I_S \rangle^2]\)^{1/2} on the single-pass gain \( G \). This quantity is equal to one according to the undepleted pump model. When pump depletion is included in the model, this quantity is seen to be equal to one for small values of \( G \) and to decrease monotonically for increasing values of \( G \). This behavior can be understood as follows: In the limit of no pump depletion, the Stokes output is due simply to the linear amplification of light scattered from thermally generated phonons. The Stokes output hence has fluctuations characteristic of thermal light, for which the normalized variance is unity. In the saturation region these fluctua-
tions tend to be damped out due to saturation of the amplification process. The curve labeled localized fluctuating source was calculated using the method shown in the Appendix; this curve shows the most rapid falloff with increasing values of $G$. The other two curves, labeled distributed fluctuating source, were obtained using the numerical technique described above. These curves show that the fluctuations in the Stokes output are reduced less dramatically through use of a small value of $\Gamma T_i$. The reason for this behavior is that the limit of small $\Gamma T_i$ corresponds to a medium for which the transit time $T_i$ is much shorter than the time scale $\Gamma^{-1}$ on which the fluctuations in the Stokes intensity occur. In such a case, the degree of pump depletion can change appreciably over the time duration of a fluctuation, and this effect tends to smooth out the fluctuations in the Stokes output intensity.\(^{16}\)

**IX. SPECTRUM OF THE STOKES RADIATION**

We define the spectral density $S(\omega)$ of the Stokes radiation as the Fourier transform of the field autocorrelation function, that is,

$$S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \tau} \langle E^*_S(O,\tau) E_S(O,0) \rangle d\tau \ .$$

(31)

Through use of expression (20) for $E_S(z,\tau)$, we find by direct integration that the spectrum is given by\(^{18}\)

$$S(\omega) = \frac{4\pi n c A (\Gamma + 1)}{nc \Lambda} \left\{ \exp \left[ \frac{G(\Gamma/2)^2}{\omega^2 + (\Gamma/2)^2} \right] - 1 \right\} .$$

(32)

In the low-gain limit ($G \ll 1$), Eq. (32) becomes

$$S(\omega) = \frac{4\pi n c A (\Gamma + 1)}{nc \Lambda} \frac{1}{\omega^2 + (\Gamma/2)^2} ,$$

(33)

which has the form of the Lorentzian line shape characteristic of spontaneous Brillouin scattering. In the high-gain limit ($G \gg 1$), Eq. (32) becomes

$$S(\omega) = \frac{4\pi n c A (\Gamma + 1)}{nc \Lambda} \exp \left[ -\left( \frac{4G\omega^2}{\Gamma^2} \right) \right] ,$$

(34)

which has the form of a Gaussian line shape. The full width at half maximum (FWHM) width of the spectrum in this limit is given by

$$\Delta \omega = \Gamma \ln(2G)^{1/2} .$$

(35)

The narrowing of the spectrum with increasing values of $G$ is known as gain narrowing. The dependence of the Stokes linewidth on the single-pass gain $G$ as predicted by the general result of Eq. (32) is shown in Fig. 6 by the curve labeled distributed fluctuating source (undepleted pump).

Also shown in Fig. 6 is the curve labeled distributed fluctuating source. This curve was obtained from the width of the power spectrum $S(\omega)$ estimated as the squared modulus of the Fourier transform of a segment of the Stokes field strength $E_S(t)$ obtained by numerical integration of the SBS equations. We see that the Stokes spectrum narrows with increasing gain $G$ less rapidly than the dependence predicted by the constant-pump theory.

**X. CONCLUSIONS**

We have developed a model that shows how SBS is initiated by spatially distributed thermal fluctuations of the density of the Brillouin medium. Under conditions near and below the threshold for SBS, pump depletion effects are negligible, and we have obtained an analytic expression showing how the SBS reflectivity depends on the single-pass gain $G$ through the medium. From this formula, we obtain an explicit prediction for the value of $G$ at the threshold. These predicted values are in good agreement with the empirically known values. We have also conducted numerical studies of the complete set of coupled equations that describe the SBS process including thermal initiation and pump depletion effect. We find that the output intensity is a fluctuating quantity and that the relative magnitude of these fluctuations is decreased through use of a large value of the single-pass gain. We have also compared the results of our model to those of the simpler model of Zel'dovich, Piliptskii, and Shkunov (which ignores the distributed nature of the initiation), and we find that the two models are in good qualitative but not quantitative agreement.

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**APPENDIX**

In this appendix, we present a generalization of the localized source model of Zel'dovich, Piliptskii, and Shkunov that allows the Stokes seed to be a stochastic quantity. Our starting point is the relation (29) between the reflectivity $R$ and the single-pass gain $G$:

$$G = \frac{\ln R + C_{\text{ref}}}{1 - R} ,$$

(A1)

where

$$C_{\text{ref}} = -\ln f$$

(A2)

and where

$$f = I_s(L)/I_s(L) .$$

(A3)

Unlike in the original treatment of Zel'dovich, Piliptskii, and Shkunov, we allow $f$ to be a stochastic quantity. We assume that the probability distribution of $f$ is that characteristic of a thermal source with mean value equal to $f_0$:

$$P(f) = \frac{1}{f_0} \exp(-f/f_0) .$$

(A4)

The fluctuating seed induces the fluctuations of the reflectivity $R$. The probability distribution of the reflectivity $R$ is given by
Through use of this expression we can compute the expectation values of various functions of the reflectivity. The curves labeled fluctuating localized source in Figs. 3 and 5 were obtained in this manner.