ANGULAR SPECTRUM REPRESENTATION

OF

ELECTROMAGNETIC FIELDS

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- SUMMARY SHEETS
- DERIVATIONS

NOTATION: ANALYTIC SIGNAL FOR FIELDS

$$e^{i\omega t} \quad \text{OR} \quad \int_{-\infty}^{\infty} - e^{-i2\pi\nu t}\, dt$$

REF: The Plane Wave Spectrum Representation of Electromagnetic Fields
P.C. Clemmow, Pergamon Press, (1966).

$$\nabla \times \underline{\mathcal{E}} = -\frac{\partial \underline{\mathcal{B}}}{\partial t} \qquad\qquad \nabla \times \mathbf{E} = -i2\pi\nu\,\mathbf{B}$$

$$\nabla^2 \mathcal{E}_y = \mu\varepsilon \frac{\partial^2 \mathcal{E}_y}{\partial t^2} \qquad\qquad \left(\nabla^2 + k^2\right) E_y(x,y,z;\nu) = 0$$

$$\begin{tikzcd}
\mathcal{E}_y(x,y,z,t) \arrow[rr, "\mathcal{F}_t", "{\int --- e^{-i2\pi\nu t}dt \;\; (\text{or htd } e^{+i2\pi\nu t})}"'] \arrow[drr, "\mathcal{F}_{xyt}"'] & & E_y(x,y,z;\nu) \arrow[d, "\mathcal{F}_{xy}"] \\
& & V_y\left(f_x, f_y; z; \nu\right)
\end{tikzcd}$$

$$\frac{\partial^2 V_y}{\partial z^2} + \left[k^2 - \left(2\pi f_x\right)^2 - \left(2\pi f_y\right)^2\right] V_y = 0$$

$$V_y\left(f_x, f_y; z; \nu\right) = \iint_{-\infty}^{\infty} dx\,dy\,E_y(x,y,z;\nu)\,e^{-i2\pi f_x x - i2\pi f_y y}$$

$$E_y(x,y,z;\nu) = \iint_{-\infty}^{\infty} df_x\,df_y\,V_y\left(f_x, f_y; z; \nu\right) e^{+i2\pi f_x x + i2\pi f_y y}$$

Signal Representations in Fourier Optics
Differential Equations

OPT 461
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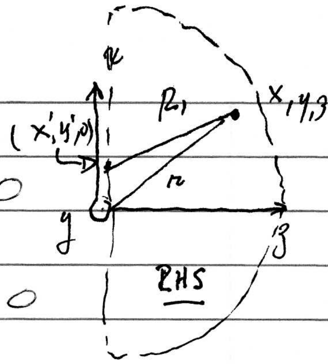
ANGULAR SPECTRUM OF PLANE WAVES



$$\nabla^2 E_y(x,y,z;\nu) + k^2 E_y(x,y,z;\nu) = 0$$

$$\nabla^2 E_{ya}(x,y,z;\nu) + k^2 E_{ya}(x,y,z;\nu) = 0$$

DENOTE THE 2-DIMENSIONAL SPATIAL FOURIER XFM: V_{ya}

$$V_{ya} = \iint E_{ya}(x,y,z;\nu)\, e^{-i2\pi f_x x - i2\pi f_y y}\, dx\, dy$$

& INVERSE

$$\frac{\partial^2 V_{ya}}{\partial z^2} + \left(k^2 - (2\pi f_x)^2 - (2\pi f_y)^2\right) V_{ya}(f_x,f_y;z;\nu) = 0$$

Completely analogous to the Telegrapher's Equation - solution is

$$V_{ya} = A\, e^{-ih_+ z} + B\, e^{+ih_+ z}$$

$$h_+ = \left[k^2 - (2\pi f_x)^2 - (2\pi f_y)^2\right]^{1/2}$$

Group → traveling waves & damped exponentials

$$-ih_+ = -i(25)^{1/2} = -i5$$

$$-ih_+ = -i\,(-36)^{1/2} = (-i)(+i6) = -6 \Rightarrow e^{-6z}$$

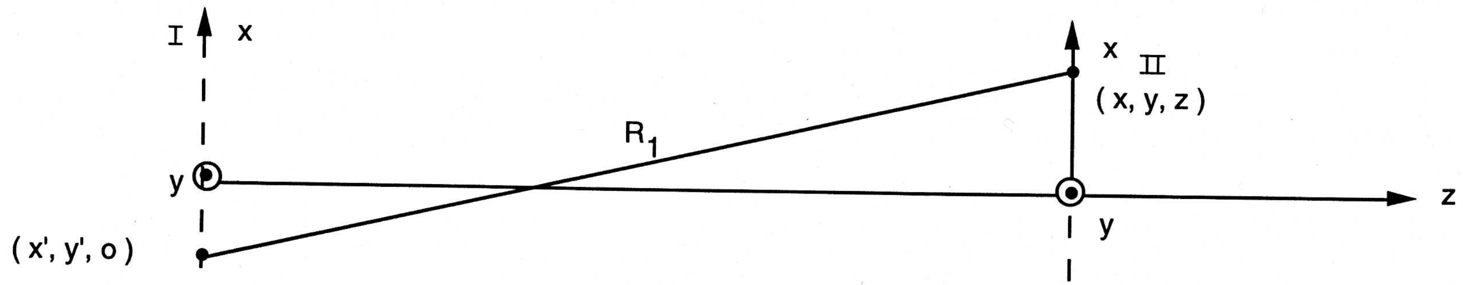
We see that the Fourier transform multiplier is given by $e^{-ih_+ z}$

$$V_{ya}(f_x,f_y;z;\nu) = V_{ya}(f_x,f_y;0;\nu)\, e^{-ih_+ z}$$

Then, obviously $e^{ih_+ z}$ has incoming waves from $+z$ and growing waves with z

We drop these for the RSS.

Rare cases - we do not drop!!



$$
\begin{tikzcd}
E_{ya}(x, y;\ o;\ \nu) \arrow[rr, "\begin{pmatrix}\text{Sommerfeld-}\\ \text{Rayleigh-}\\ \text{Smythe}\\ \text{Integral}\end{pmatrix}"] \arrow[d, "\mathcal{F}_{xy}"'] & & E_{ya}(x, y;\ z;\ \nu) \arrow[d, "\mathcal{F}_{xy}"] \\
V_{ya}(f_x, f_y; o; \nu) \arrow[rr, dashed, "e^{-i 2\pi z \pm\sqrt{\frac{1}{\lambda^2} - f_x^2 - f_y^2}}"] & & V_{ya}(f_x, f_y; z; \nu)
\end{tikzcd}
$$

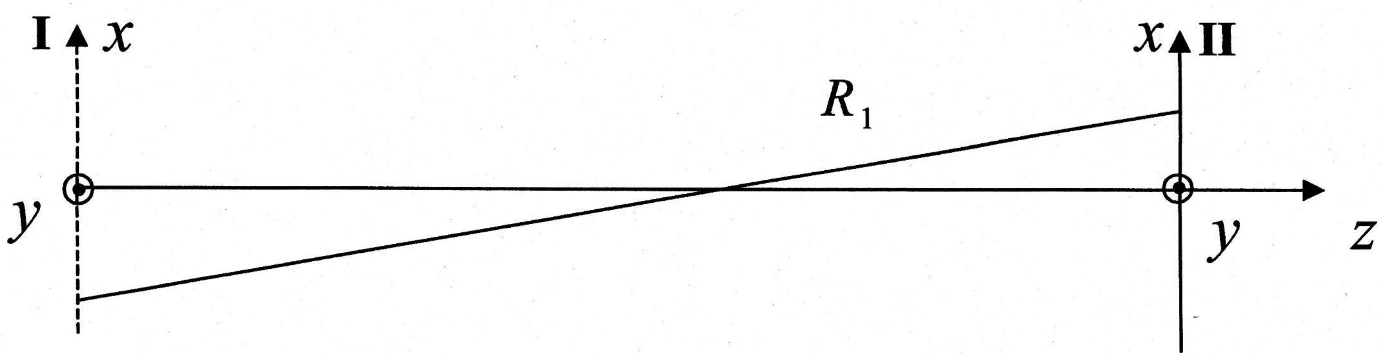
$$E_{ya}(x, y;\ z;\ \nu) = \int_{-\infty}^{\infty}\!\!\int dx'dy'\, E_{ya}(x', y', o; \nu)\, \frac{e^{-ikR_1}}{2\pi R_1}\left(ik + \frac{1}{R_1}\right)\frac{z}{R_1}$$

$$V_{ya}(f_x, f_y; z; \nu) = V_{ya}(f_x, f_y; o; \nu)\, e^{-i 2\pi z \pm\sqrt{\frac{1}{\lambda^2} - f_x^2 - f_y^2}}$$

$$\frac{e^{-ikR_1}}{2\pi R_1}\left(ik + \frac{1}{R_1}\right)\frac{z}{R_1} = \int_{-\infty}^{\infty}\!\!\int df_x\, df_y\, e^{i 2\pi\left[f_x(x - x') + f_y(y - y') - z\pm\sqrt{\frac{1}{\lambda^2} - f_x^2 - f_y^2}\right]}$$

Angular Spectrum Formulas in an Analytic Signal Representation
Radiation from a Planar Aperture

OPT 461 (1998)
Nicholas George



$$
\begin{tikzcd}[column sep=huge, row sep=huge]
E_{ya}(x, y, 0;\nu) \arrow[r, shift left, "{\int\limits_{-\infty}^{\infty}\int dx'dy'\ E_{ya}(x', y', 0;\nu)\ \frac{e^{-ikR_1}}{2\pi R_1}(ik+\frac{1}{R_1})\frac{z}{R_1}}"] \arrow[d, "F_{xy}"] & E_{ya}(x, y, z;\nu) \arrow[l, shift left, "{\int\limits_{-\infty}^{\infty}\int dx\,dy\ E_{ya}(x, y, z;\nu)\ \frac{e^{ikR_1}}{2\pi R_1}(-ik+\frac{1}{R_1})\frac{z}{R_1}}"] \arrow[d, "F_{xy}"] \\
V_{ya}(f_x, f_y;0;\nu) \arrow[r, shift left, "{V_{ya}(f_x,f_y;0;\nu)\exp[-iz\mp\sqrt{k^2-4\pi^2(f_x^2+f_y^2)}\ ]}"] & V_{ya}(f_x, f_y;z;\nu) \arrow[l, shift left, "{V_{ya}(f_x,f_y;z;\nu)\exp[\ iz\mp\sqrt{k^2-4\pi^2(f_x^2+f_y^2)}\ ]}"]
\end{tikzcd}
$$

Fig. A : Exact solutions for inverse problems in EM theory

$\boxed{\text{ANGULAR SPECTRUM REPRESENTATION : GENERAL SOLUTION}}$

ANGULAR SPECTRUM 20 April 2005

Lecture Outline : 1) DEFINITION $E_y(x,y,z,t) \to E_y(x,y,z;\nu)$

2) Why the name $\downarrow$

In the inversion process $\boxed{V_y(f_x,f_y,z;\nu)}$

$$\Delta E_y(x,y,z;\nu) = V_y(f_x,f_y;z,\nu)e^{+i2\pi f_x x+i2\pi f_y y}\Delta f_x \Delta f_y$$

$e^{i\omega t-i\underline{k}\cdot\underline{r}}$ IS PLANE WAVE $A\, e^{-(ik_x x+ik_y y+ik_z z)}$

$$k_z = \left[k^2-(2\pi f_x)^2-(2\pi f_y)^2\right]^{1/2}$$

3) So - CONSIDER

$$E_y(x,y,z;\nu) = \iint_{Apert.} E_y(x,y;0;\nu)\underbrace{\frac{e^{-ik\sqrt{(x-x')^2+(y-y')^2+(z-0)^2}}}{2\pi R_1}\frac{z}{R_1}\left(ik+\frac{1}{R_1}\right)}_{h \text{ or } ~~p~~ \text{ or } b}dx'dy'dz'$$

PERFECT CONVOLUTION

$$E_y(x,y,z;\nu = E_y(x,y,0) ** h(x,y)$$

$$h(x,y) = \frac{e^{-ik\sqrt{x^2+y^2+z^2}}}{2\pi r}\frac{z}{r}\left(ik+\frac{1}{r}\right)$$

As typical in Fourier optics two ways to do this

We need to compute $H(f_x,f_y)$ in order to fill-in table

INPUT	IR	TF
amplitude	$h(x,y)$ $\xrightarrow{\mathcal{F}_{xy}}$	$H(f_x,f_y)$
incoherent illum	hh^* $\xrightarrow{\mathcal{F}_{xy}}$	$H \star H$ NORMALIZED FOR OTF

A DIRECT DERIVATION & A NEW FT RESULT

Looking at the Rayleigh-Sommerfeld-Smythe formula

$$E_x(x,y,z,\nu) = \iint_{-\infty}^{\infty} E_x(x',y',0;\nu)\,\frac{e^{-ikR_1}}{2\pi R_1}\,\frac{z}{R_1}\left(ik + \frac{1}{R_1}\right)dx'dy'$$

we see that

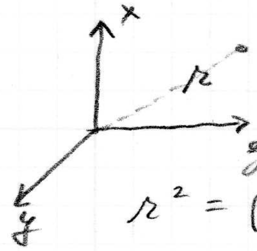
$$\mathcal{F}_{xy}\,\frac{e^{-ik(x^2+y^2+z^2)^{1/2}}}{2\pi r}\,\frac{z}{r}\left(ik + \frac{1}{r}\right) = e^{-iz\sqrt{k^2-(2\pi f_x)^2-(2\pi f_y)^2}}$$

Now, it is interesting and challenging to derive this Fourier transform directly, which we do on the next couple of pages —

CONSIDER THE DIRECT $\mathcal{F}$ OF

$$\frac{e^{-ikr}}{r}$$

where $r^2 = x^2+y^2+z^2$



$r^2 = \rho^2 + z^2$

$$F = \iint_0^\infty \frac{e^{-ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}\, e^{-i2\pi f_x x - i2\pi f_y y}\, dx\, dy$$

$f_x = f_\rho \cos\gamma$

$f_y = f_\rho \sin\gamma$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$$F = \iint \frac{e^{-ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}\, e^{-i2\pi f_\rho \rho[\cos\phi\cos\gamma + \sin\phi\sin\gamma]}\, \rho\, d\rho\, d\phi$$

$$\int_0^{2\pi} e^{-i2\pi f_\rho \rho\cos(\phi-\gamma)}\, d\phi = 2\pi J_0(2\pi f_\rho \rho)$$

$$F = 2\pi \int_0^\infty \frac{J_0(2\pi f_\rho \rho)\, e^{-ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}\, \rho\, d\rho$$

Let $u^2 = \rho^2 + z^2$

$\cancel{2}u\,du = \cancel{2}\rho\,d\rho$

$$F = 2\pi \int_0^\infty J_0\left(2\pi f_\rho \sqrt{u^2-z^2}\right) \frac{e^{-iku}}{\cancel{u}}\, \cancel{u}\, du$$

Like a Laplace transform of J_0

$$F = 2\pi\left(\int_0^\infty J_0\big(\underbrace{2\pi f_\rho}_{=b} \sqrt{u^2-z^2}\big)\cos(\underbrace{k}_{=c} u)\,du - i\int J_0\left(2\pi f_\rho\sqrt{u^2-z^2}\right)\sin(ku)\,du\right)$$

G+R 6.677.2

$k^2 - (2\pi f_x)^2 - (2\pi f_y)^2 > 0$

$k > 2\pi f_\rho$

$c > b$.

G+R 6.677.1

$-(\sin + i\cos)$

$-i\left(\frac{\sin}{i} + \cos\right)$

$-i(-i\sin + \cos)$

$$F = 2\pi\left(-\frac{\sin\left(z\sqrt{k^2-(2\pi f_\rho)^2}\right)}{\sqrt{k^2-(2\pi f_\rho)^2}} - i\,\frac{\cos z\sqrt{k^2-(2\pi f_\rho)^2}}{\sqrt{k^2-(2\pi f_\rho)^2}}\right)$$

$$F = -2\pi i\left[\cos(\,) - i\sin(\,)\right] = -2\pi i\,\frac{e^{-iz\sqrt{k^2-(2\pi f_\rho)^2}}}{\sqrt{k^2-(2\pi f_\rho)^2}}$$

Consistent with

2.120 Clemmow

$$\frac{\partial}{\partial z}\frac{e^{-ikr}}{2\pi r} = \frac{e^{-ikr}}{2\pi r}\left[-ik - \frac{1}{r}\right]\frac{\partial r}{\partial z}$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial z} = 2z$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial}{\partial z}\left(\frac{e^{-ikr}}{2\pi r}\right) = -\frac{e^{-ikr}}{2\pi r}\left(ik + \frac{1}{r}\right)\frac{z}{r}$$

$$E_x(x,y,z) = \iint E_x(x',y',0;\nu)\,\frac{e^{-ik\left[(x-x')^2+(y-y')^2+z^2\right]^{1/2}}}{2\pi R_1}\,\frac{z}{R_1}\left(ik + \frac{1}{R_1}\right)dx'dy'$$

Impulse response is perfect convolution with

$$p(x,y;z;\nu) = \frac{e^{-ik(x^2+y^2+z^2)^{1/2}}}{2\pi r}\,\frac{z}{r}\left(ik + \frac{1}{r}\right)$$

$$P(f_x,f_y;z;\nu) = \iint p(x,y;z;\nu)\,e^{-i2\pi f_x x - i2\pi f_y y}\,dxdy$$

$$P = \mathcal{F}_{xy}\,\frac{e^{-ikr}}{2\pi r}\,\frac{z}{r}\left(ik + \frac{1}{r}\right) = -\mathcal{F}_{xy}\,\frac{\partial}{\partial z}\left(\frac{e^{-ikr}}{2\pi r}\right) = \frac{\partial}{\partial z}\mathcal{F}_{xy}\ldots$$

$$P = +\frac{2\pi i}{2\pi}\,\frac{\partial}{\partial z}\,\frac{e^{-iz\sqrt{k^2-(2\pi f_\rho)^2}}}{\sqrt{k^2-(2\pi f_\rho)^2}} \qquad \Bigg\} \quad \frac{i(-i)\sqrt{\ }}{\sqrt{\ }} = +1$$

$$P = e^{-iz\left[k^2-(2\pi f_\rho)^2\right]^{1/2}}$$

$$P(f_x,f_y;z;\nu) = e^{-i2\pi z\left[\left(\frac{1}{\lambda}\right)^2-(f_x^2+f_y^2)\right]^{1/2}}$$

QED

$$V_{ya}(f_x, f_y; z; \nu) = V_{ya}(f_x, f_y; 0; \nu)\, e^{-i2\pi z\left[\frac{1}{\lambda^2} - f_x^2 - f_y^2\right]^{1/2}}$$

$$\downarrow \mathcal{F}^{-1}$$

$$E_{ya}(x, y, z, \nu) = \iint_{-\infty}^{\infty} V_{ya}(f_x, f_y; 0; \nu)\, e^{i2\pi f_x x + i2\pi f_y y - i2\pi z\left[\ \right]^{1/2}}\, df_x\, df_y \qquad R_1^2 = (x-x')^2 + (y-y')^2 + z^2$$

$$= \iiiint_{-\infty}^{\infty} E_{ya}(x', y'; 0; \nu)\, e^{-i2\pi f_x x' - i2\pi f_y y'}\, e^{i2\pi f_x x + i2\pi f_y y - i2\pi z\left[\ \right]^{1/2}}\, df_x\, df_y\, dx'\, dy'$$

$$= \iint_{-\infty}^{\infty} E_{ya}(x', y'; 0; \nu)\, e^{i2\pi f_x (x-x') + i2\pi f_y (y-y') - i2\pi z\left[\ \right]^{1/2}}\, df_x\, df_y\, dx'\, dy'$$

$$= \iint_{-\infty}^{\infty} E_{ya}(x', y'; 0; \nu)\, \frac{e^{-ikR_1}}{2\pi R_1}\, \frac{z}{R_1}\left(ik + \frac{1}{R_1}\right) dx'\, dy'$$

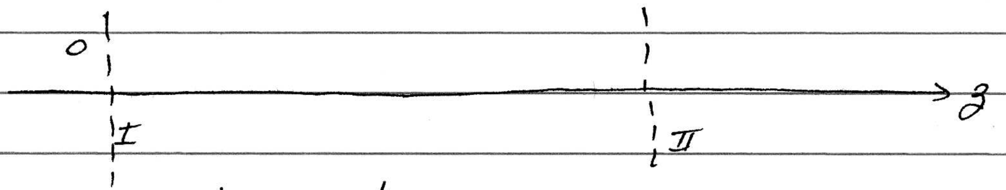
$$\therefore \frac{e^{-ikR_1}}{2\pi R_1}\, \frac{z}{R_1}\left(ik + \frac{1}{R_1}\right) = \iint_{-\infty}^{\infty} e^{i2\pi f_x (x-x') + i2\pi f_y (y-y') - i2\pi z\left[\frac{1}{\lambda^2} - f_x^2 - f_y^2\right]^{1/2}}\, df_x\, df_y$$

ILLUSTRATION - ANGULAR SPECTRUM

N. George

Consider a plane wave incident on $z=0$ plane - Its form is given by

$$E_0 \, e^{-i\underline{k}\cdot\underline{r}} \qquad -i\underline{k}\cdot\underline{r} = -i k_{x_0} x - i k_{y_0} y - i k_{z_0} z$$



$$E \qquad E_0 \, e^{-i 2\pi \frac{k_{x0}}{2\pi} x - i 2\pi \frac{k_{y_0}}{2\pi} y}$$

$$\Big\downarrow \mathcal{F}_{xy}$$

$$\int_{-\infty}^{\infty} e^{-i 2\pi \frac{k_{x0}}{2\pi} x - i 2\pi f_x x} \, dx \quad \cdots \qquad V(f_x, f_y; 0) \, e^{-iz 2\pi \sqrt{\frac{1}{\lambda^2} - f_x^2 - f_y^2}}$$

$$E_0 \, \delta\!\left(f_x + \frac{k_{x0}}{2\pi}\right) \delta\!\left(f_y + \frac{k_{y0}}{2\pi}\right) \xrightarrow{\text{PRODUCT}} E_0 \, \delta\!\left(f_x + \frac{k_{x0}}{2\pi}\right) \delta\!\left(f_y + \frac{k_{y0}}{2\pi}\right) \times e^{-i 2\pi z \sqrt{}}$$

INVERT AT PLANE II FOR

$$E_{OUT} = \iint_{-\infty}^{\infty} E_0 \, \underbrace{\delta\!\left(f_x + \frac{k_{x0}}{2\pi}\right)}_{0} \underbrace{\delta\!\left(f_y + \frac{k_{y0}}{2\pi}\right)}_{0} e^{-i 2\pi z \sqrt{}} \cdot e^{i 2\pi f_x x + i 2\pi f_y y} \, df_x \, df_y$$

$$= E_0 \, e^{-i 2\pi f_{x0} x + i 2\pi f_{y_0} y - i z \left[k^2 - 2\pi\left(f_{x0}^2 + f_{y_0}^2\right)\right]^{1/2}}$$

$$= E_0 \, e^{-i 2\pi f_{x_0} x - i 2\pi f_{y_0} y - i 2\pi f_{z_0} z} \qquad \text{QED}$$