Laboratory 1: Entanglement and Bell’s Inequalities

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Abstract

The purpose of this experiment was to observe and verify quantum entanglement of the polarization states in photon pairs. The entanglement was accomplished through a nonlinear process called spontaneous parametric down conversion. To verify that entanglement was achieved, it was necessary to use a modified version of Bell’s inequalities (Clauser, Horne, Shimony, and Holt). If this classical argument is violated, then entanglement is achieved. Using Avalanche Photodiodes (APDs), the necessary measurements were taken to calculate the $|S|$ values of the adapted inequality. Having achieved an $|S|$ value of over 2, entanglement was verified for this experimental setup.

Background and Theory

Entanglement is a special feature of quantum mechanics. It is known that one can never simultaneously know with exact certainty the position or momentum of a particle. This is in accordance with Heisenberg’s Uncertainty Principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \text{ eq}(1)$$

Where $\Delta x$ is uncertainty of position, $\Delta p$ is uncertainty of momentum, and $\hbar$ is Planck’s constant, $\hbar$, divided by $2\pi$. In Heisenberg’s original version it was $\Delta x \Delta p = \hbar$ [1]. Therefore the only appropriate way to represent a particle, such as a photon, is with a wave function that represents the probability distribution of where the particle could be. It is also interesting to consider that one has no idea what a particle could be doing while no one is looking. Observing a particle collapses the wave function because there is certainty of its location and/or other properties, as opposed to a probability of this information. Until the wave function is collapsed, though, many properties of a particle are indefinite and are only definite when observed. If two particles interact, they may become entangled, meaning that they become one inseparable state, since their properties are indefinite. If one particle is observed, then reliable information is known about the other. The concept of entanglement was introduced by Einstein, Podolsky, and Rozen [2] and was developed further by Schrödinger who coined this term [3]. In Schrödinger’s 1935 paper, he wrote an example of entanglement involving a cat and a radioactive atom. The thought experiment is set up such that there is a cat in a box. In the box with the cat there is a radioactive atom, and a device that releases poison if the atom decays. When we close the box, we are uncertain if the atom has decayed or not, it is represented by a superposition of probabilities of the state of being decayed and not being decayed (see eq(A1)). The cat is entangled with this process and its state cannot be separated from the radioactive material it is entangled with. In this paradox, the cat is both dead and alive because of the superposition of states, it is entangled with the indefinite radioactive

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atom. The example was meant to be absurd, because Schrödinger was skeptical of this interpretation as were Einstein, Podolsky, and Rozen. They thought this idea must be wrong, and it was a symptom of quantum theory’s incompletion. Entangled items can only be represented by the same wave function and cannot be separated. If one were to observe one of the particles then its wave function and any of its entangled particles’ wave functions would collapse since they are the same state. This would give reliable information about any other particle entangled with the observed particle, no matter the distance or coordinate system (see eq(A5))!

Another important theoretical matter that is vital to the experiment is the Bell inequality. The Bell inequality we used is based off of a simple classical argument that \(|a| + |b| + |c|\) is greater than or equal to \(|a + b + c|\). The only way for any Bell inequality to be violated is for a non-classical phenomenon to be present. If one can create an experiment involving entangled particles and violate the inequality, then the entanglement phenomena can be verified. The modified version of Bell’s inequalities that allows for this experimental testing is often referred to as a CHSH inequality (developed by Clauser, Horne, Shimony, and Holt) [4]. The equality is calculated as such:

\[
|S| = |E(\alpha, \beta) - E(\alpha, \beta')| + |E(\alpha', \beta) + E(\alpha', \beta')| \quad \text{eq}(2)
\]

Where

\[
E(\alpha, \beta) = \frac{N(\alpha, \beta) + N(\alpha_{\perp}, \beta_{\perp}) - N(\alpha, \beta_{\perp}) - N(\alpha_{\perp}, \beta)}{N(\alpha, \beta) + N(\alpha_{\perp}, \beta_{\perp}) + N(\alpha, \beta_{\perp}) + N(\alpha_{\perp}, \beta)} \quad \text{eq}(3)
\]

Where \(N(\alpha, \beta)\) is the number of coincidence counts when the two experimental polarizers are set to angle \(\alpha\) and angle \(\beta\) (see figure 1). Coincidence counts are the number of arrivals of single photons simultaneously to two separate detectors in a given time interval. This inequality can be the same for classical and quantum systems for certain angles, that is why it is important to use angles that optimize the difference between the classical result and the quantum result, as specified below in the experiment section.

**Experiment**

The experimental setup used utilized spontaneous parametric down conversion in conjunction with a modified version of Bell’s inequality to demonstrate entanglement of polarization states of photons.

![Figure 1: The unfolded experimental setup. The setup had a quartz plate to compensate for the phase difference generated by the different optical path lengths between the two BBO crystals. The APDs in conjunction with the](image-url)
The computer counting card counts the “coincidences,” simultaneous arrivals of photons.

The setup utilizes spontaneous parametric down conversion in a uniaxial crystal (beta barium borate). The process only converts with an efficiency on the order of $10^{-10}$. Therefore, a high power 100mW laser was used to create as many down converted photons as possible. The spontaneous parametric down conversion process converts an incident photon into two photons of longer wavelength called the signal and idler photon, with conservation of energy and momentum (see appendix for equations). In this experimental setup, the filters used transmit a very narrow band of wavelengths. The filters were selected to transmit photons with double the wavelength of the laser used (Argon ion 363.8nm). This allows only the signal and idler photons with the same wavelength (727.6nm) to be detected. The crystal we used had a type I cut, meaning the signal and idler photons’ polarizations were orthogonal to the incident photon’s polarization. By cutting the crystal such that the optical axis was at a certain angle, the incident photons and down converted photons see the same refractive index, since different polarization states will see different refractive indices in a birefringence material. This “phase matching” is what determines the output polarization states of the crossed BBO crystals. The photons incident on the crystal component had a linear polarization that was 45 degrees relative to the first crystals optical axis. This meant half of the down converted photons were down converted in each of the crystals, which are crossed by 90 degrees. Therefore there are equal numbers of photons in the “cones of light”, for the horizontal and vertical cones. Overlapping the cones we are interested in is done by using a quartz plate to add phase to the lagging cone (see eq(A2), eq(A3), and eq(A4)).

Using this nonlinear crystal scheme, which produces the signal and idler photons, the polarization entangled state can be created. Bell’s inequalities can be calculated at different polarizer settings. Using polarizers in front of single photon counting detectors as seen in figure 1, coincidence counts can be measured. When the polarizers are at parallel angles, the signal photon and the idler photon will both pass through, since they are represented by the same polarization state. If the polarizers are at perpendicular angles, then only one of the photons from the entangled photon pair can pass, while the other will be blocked, resulting in the minimum photon coincidence count. Theoretically there should be a cosine squared dependence of coincidence counts on the difference between polarization angles.

**Procedure**

1. Make sure the cone of light is properly aligned within the setup by viewing a circle of light with an EM-CCD. Place camera after the BBO crystals, insert imaging lens, and block the unconverted beam. Make sure that the ring of light that is to be observed is incident upon the detectors.
2. Check to see if fringe visibility of photon coincidence counts exceeds 0.71, in order to violate Bell’s Inequality. Visibility=$(N_{\text{max}}-N_{\text{min}})/(N_{\text{max}}+N_{\text{min}})$. If the threshold is not met there may be misalignment.
3. Check to see if the singles (non-coincidence) counts for each APD is independent of its accompanying polarizer angle. If there is a dependence on polarizer angle, there is misalignment and the quartz plate must be adjusted.
4. Sweep through a range of quartz plate angles, and for each quartz plate angle measure single counts for different polarizer angles. Plot the singles counts as a function of quartz plate angle to get a curve for each polarizer angle. Singles counts should be constant and independent of polarizer angle in this experiment, so the best intersection of the curves is the desirable quartz plate setting. This test should be performed with rotations of the quarter wave plate relative to both the vertical and horizontal axes.

5. Once sufficient alignment has been achieved, and visibility of coincidence counts is greater than or equal to 0.71, measure for all combinations of the polarizer angles $\alpha=\{0,45,90,135\}$ and $\beta=\{22.5,67.5,112.5,157.5\}$. A total of 16 measurements will be used to calculate the CHSH inequality, and verify entanglement.

**Results and Analysis**

The light cone should appear constant under rotations of a polarizer, since the two components, vertically and horizontally polarized light, are equal.

![Figure 2](image)

**Figure 2**: This figure shows the light “cone” with a filter and polarizer in front of the camera at 3 different angles. The filter only transmits a narrow bandwidth around the desired wavelength. The light cone is constant under the rotation because the vertical and horizontal cones are equal and overlapping, making this an un-polarized light cone.

Initially checking for fringe visibility of the coincidence counts, it was below the 0.71 threshold at 0.54. Adjustment of the quartz plate was necessary. Alignment of the quartz plate is difficult because there are two axes about which is can be rotated. Both axes were rotated through a small range of angles to see if the best alignment could be determined.
**Figure 3:** The figure shows the result of scanning through a small range of angles for the quartz plate vertical axis, at different polarizer angles. Both polarizers were set to the same setting for maximum coincidence count. It is desirable to reach the best intersection, so that maximum coincidence counts are not dependent on the parallel polarizers’ angle.

**Figure 4:** The figure shows the result of scanning through a small range of angles for the quartz plate horizontal axis, at different polarizer angles. Both polarizers were set to the same setting for maximum coincidence count. It is desirable to reach the best intersection, so that maximum coincidence counts are not dependent on the parallel polarizers’ angle.
After the preliminary alignment was performed, the data for the CHSH inequality was taken for the first attempt at violating Bell’s inequality. The quartz plate was set to 35 degrees vertical and -0.5 horizontal.

<table>
<thead>
<tr>
<th>Polarizer A</th>
<th>Polarizer B</th>
<th>Net coincidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>-45</td>
<td>-22.5</td>
<td>241.6004625</td>
</tr>
<tr>
<td>-45</td>
<td>22.5</td>
<td>49.70820696</td>
</tr>
<tr>
<td>-45</td>
<td>67.5</td>
<td>27.81832161</td>
</tr>
<tr>
<td>-45</td>
<td>112.5</td>
<td>240.8037829</td>
</tr>
<tr>
<td>0</td>
<td>-22.5</td>
<td>200.9381061</td>
</tr>
<tr>
<td>0</td>
<td>22.5</td>
<td>235.7104968</td>
</tr>
<tr>
<td>0</td>
<td>67.5</td>
<td>128.2238311</td>
</tr>
<tr>
<td>0</td>
<td>112.5</td>
<td>84.34675305</td>
</tr>
<tr>
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<td>34.29263416</td>
</tr>
<tr>
<td>45</td>
<td>22.5</td>
<td>287.6417333</td>
</tr>
<tr>
<td>45</td>
<td>67.5</td>
<td>314.515383</td>
</tr>
<tr>
<td>45</td>
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<td>74.17161817</td>
</tr>
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</tr>
<tr>
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<td>22.5</td>
<td>96.9416073</td>
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<td>90</td>
<td>67.5</td>
<td>214.7333936</td>
</tr>
<tr>
<td>90</td>
<td>112.5</td>
<td>209.8243169</td>
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</table>

**Table 1:** The specific polarizer angles that are meant to maximize the difference between quantum and classical results. The coincidence counts corresponding to the combinations of these angles are used to calculate Bell’s Inequality.

Using the values in table 1 and equations 2 and 3, the CHSH inequality was calculated to be 2.15. Since this values was greater than two, the inequality has been violated and entanglement verified. However, the best fringe visibility was only 0.62 at some polarizer angle. Further alignment was necessary.

<table>
<thead>
<tr>
<th>Polarizer A</th>
<th>Polarizer B</th>
<th>Net coincidence</th>
<th>Visibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>179.708</td>
<td>0.486652</td>
</tr>
<tr>
<td>0</td>
<td>90</td>
<td>62.05404</td>
<td>0.575018</td>
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<tr>
<td>90</td>
<td>90</td>
<td>229.97775</td>
<td>0.976704</td>
</tr>
<tr>
<td>45</td>
<td>45</td>
<td>306.3047</td>
<td>3.609865</td>
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<td>288.2872</td>
<td>0.975266</td>
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<td>60</td>
<td>60</td>
<td>298.8474</td>
<td>0.689563</td>
</tr>
<tr>
<td>60</td>
<td>150</td>
<td>17.46639</td>
<td>0.864092</td>
</tr>
<tr>
<td>150</td>
<td>150</td>
<td>239.566</td>
<td>0.864092</td>
</tr>
</tbody>
</table>

**Table 2:** The data points in this table are to determine the fringe visibility of coincidence counts for the quartz plate vertical angle of 35 degrees.
<table>
<thead>
<tr>
<th>Polarizer A</th>
<th>Polarizer B</th>
<th>Net coincidence</th>
<th>Visibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>168.0144</td>
<td>0.507352</td>
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<tr>
<td>0</td>
<td>90</td>
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<td>90</td>
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<td>45</td>
<td>348.4706</td>
<td>0.990587</td>
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<td>45</td>
<td>135</td>
<td>1.647833</td>
<td></td>
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<tr>
<td>135</td>
<td>135</td>
<td>208.321</td>
<td>0.984304</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>341.3901</td>
<td>0.920388</td>
</tr>
<tr>
<td>60</td>
<td>150</td>
<td>14.15272</td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>150</td>
<td>187.312</td>
<td>0.859502</td>
</tr>
</tbody>
</table>

**Table 3:** The data points in this table are to determine the fringe visibility of coincidence counts for the quartz plate vertical angle of 34 degrees.

As can be seen from tables 2 and 3, a better quartz plate angle to use is 34 degrees vertical, for better fringe visibility. A test for cosine squared dependence of coincidence counts on the difference in polarizer angles was then performed.

**Coincidence Counts**

**Figure 5:** The cosine squared dependence is very clear in this graph. However, it appears that there is still misalignment. The two curves are differing in magnitude by a factor of 2.

Further alignment was carried out in the same manner as previously described yielding a more desirable magnitude in the cosine squared dependence curves.
Figure 6: Further alignment made the coincidence count dependences more equal for the horizontal and vertical components. The polarizer A angles were perpendicular between the two curves to demonstrate this. The best alignment achieved yielded a fringe visibility of 0.99, and an $S$ value of 2.30, having finally met the threshold visibility, it is more certain that Bell’s inequality is violated and entanglement is present. Without polarizer A in place, any photon picked up by the APD corresponding to polarizer B that has an entangled idler will be counted as a coincidence. Since, the idler of interest in this case will make it to the APD corresponding to polarizer A because there is no polarizer A in place. This was confirmed by removing polarizer A and measuring many angles of B. A flat curve further indicates good alignment.

Figure 7: This figure shows that coincidence counts are constant while polarizer B is being rotated without polarizer A in place.

In order to demonstrate the importance of measurement angles in the calculation of the Bell inequality, arbitrary angles were used to calculate the inequality.
Table 4: This table has combinations of arbitrary angles and their coincidence counts.

From table 4, the S value is calculated to be 0.35, which does not violate Bell’s inequality. This information is useless in terms of verifying quantum phenomena. This demonstrates the importance of using the specific angles needed to violate the inequality.

### Conclusion

This experiment demonstrated polarization state entanglement in photon pairs. The entanglement was verified by use of the CHSH Bell inequality. Alignment was an issue that impeded the results in the beginning. Once sufficient alignment was achieved, the S values reached a maximum of 2.30. Alignment on this experimental setup is crucial. Alignment is also a very difficult task.

### References


## Appendix

Entangled particles $\alpha$ and $\beta$ cannot be separated into separate states [5]:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |V_a V_{\beta}\rangle + |H_a H_{\beta}\rangle \right) \quad \text{eq}(A1)$$

The parametric down conversion process creates two new photons, from one photon, in the first BBO crystal:

$$|H\rangle \rightarrow |V_a V_i\rangle \quad \text{eq}(A2)$$

The second crystal converts photons of the other polarization components of incident beam:

$$|V\rangle \rightarrow |H_s H_i\rangle \quad \text{eq}(A3)$$

When overlapping the two components, there is a phase term, due to the difference in optical path lengths [5]. This is why phase matching is necessary:

$$|\psi_{\text{entangled}}\rangle = \frac{1}{\sqrt{2}} \left( |V_a V_i\rangle + e^{i\phi} |H_s H_i\rangle \right) \quad \text{eq}(A4)$$

It can be demonstrated that the entanglement state is independent of coordinate system, by rotating the system by an arbitrary angle $\alpha$:

$$|H\rangle = \cos \alpha |H_\alpha\rangle - \sin \alpha |V_\alpha\rangle \quad |V\rangle = \cos \alpha |V_\alpha\rangle + \sin \alpha |H_\alpha\rangle$$

Then,

$$|\psi_{\text{entangled}}\rangle = \frac{1}{\sqrt{2}} \left( [\cos \alpha |V_\alpha\rangle_s + \sin \alpha |H_\alpha\rangle_s] [\cos \alpha |V_\alpha\rangle_i + \sin \alpha |H_\alpha\rangle_i] \right)$$

$$|\psi_{\text{entangled}}\rangle = \frac{1}{\sqrt{2}} \left( (\cos^2 \alpha + \sin^2 \alpha) |V_\alpha\rangle_s |V_\alpha\rangle_i + (\cos^2 \alpha + \sin^2 \alpha) |H_\alpha\rangle_s |H_\alpha\rangle_i \right)$$

$$|\psi_{\text{entangled}}\rangle = \frac{1}{\sqrt{2}} \left( |V_\alpha\rangle_s |V_\alpha\rangle_i + |H_\alpha\rangle_s |H_\alpha\rangle_i \right) \quad \text{eq}(A5)$$