Entanglement and Bell’s Inequalities

Benjamin Feifke, Kara Morse

Professor Svetlana Lukishova

Abstract

The purpose of this experiment was to observe quantum entanglement by calculating Bell’s Inequality for entangled polarization states of photons. Photons with entangled polarization states were created by spontaneous parametric down conversion through two Type I BBO crystals. Individual photons were detected by two avalanche photodiodes (APDs) and coincidence counts of the two detectors were measured for different polarizer orientations. We measured a value of greater than 2 for Bell’s Inequality, which violates Bell’s Inequality, verifying that we had produced entangled photons.

Theory and Background

Quantum entanglement is often considered to be one of the ‘weirdest’ things in the field of quantum mechanics. Einstein called it “spooky action at a distance”. It has many implications for quantum communication and quantum teleportation, among other things.

Quantum entanglement is when the combined state (this can mean any innate characteristic of a photon, such as spin, or in this experiment, polarization) of two particles cannot be mathematically separated into two individual states. The result of this is that any action performed on one particle (also called ‘signal’ particle) of the entangled pair can be observed in the other particle (also called ‘idler’ particle), regardless of the distance between these particles.

There are a few ways to create photons with entangled states. In this experiment, photons from a laser pump are sent through two sequentially placed Type I BBO (barium borate) crystals. The first BBO crystal is oriented at an angle $\alpha$, and the second BBO crystal is oriented to the same angle, plus an additional 90 degrees. BBO crystals are nonlinear and have a strong uniaxial birefringence, so, when an incident photon that is horizontally (or vertically) polarized passes through, two entangled photons—signal and idler—are produced by spontaneous parametric down conversion, each with exactly half of the incident photon’s energy and vertically (or horizontally) polarized. These entangled photons propagate outward at the same angle from the axis of the laser pump, creating a cone of ‘entangled light’. Therefore, two cones are produced—one of horizontally polarized light, and one of vertically polarized light. (It is necessary to note that only a very small amount of photons, on the order of one out of every $10^{10}$, experience spontaneous parametric down conversion—the rest travel through the crystals and hit the beam stop.) The APDs can be placed at the intersection ‘axes’ of these two cones of light to detect entangled behavior. Despite each cone being polarized individually, the intersection axes of the two cones contain an even amount of both horizontally and vertically polarized photons.
(Figure 1). Placing linear polarizers between the crystals and the APDs allows us to manipulate the polarization of the light entering one APD or the other. The number of photons that hit the APDs at the same time with the same polarization can be measured—this is the coincidence count, and indicates that entangled photons have been produced.

Figure 1:

The first BBO crystal in the experimental setup produces entangled vertically polarized photons from incident horizontally polarized photons, while the incident vertically polarized photons pass straight through. The result of this is that the entangled vertically polarized photons are now traveling at an angle to the horizontal axis, and therefore travel a longer optical path distance in the second crystal than the incident vertically polarized photons. The result of this is a phase difference called a birefringent delay. This phase difference can be compensated for (i.e. made equal to zero) by adding a quartz plate which has a different index of refraction depending on the polarization of the incident light, thereby canceling out the phase difference induced by the BBO crystals.

Figure 2:

---

2 OPT253 lab 1 manual
There are two ways to verify that we have entanglement. The first involves using the polarizers between the two BBO crystals and the APDs. By keeping one polarizer constant and altering the angle of the second polarizer, we would expect to see a cosine squared relationship between the angle and the coincidence count in accordance with Malus’s Law. We can plot this relationship—angle for x-axis, coincidence count for y-axis—and calculate fringe visibility of the coincidence counts. A value of at least .71 for fringe visibility would verify entanglement.

The second method for verifying that we have entanglement is through the use of Bell’s Inequality. Bell’s Inequality is a classical mathematical relationship that is derived from the mathematical identity that la+b+c is less than or equal to lal + lbl + lcl—violating it would mean that we have non-classical behavior which, in this case, would mean entanglement. The equation is

$$|S| = |E(\alpha, \beta) - E(\alpha, \beta')| + |E(\alpha', \beta) - E(\alpha', \beta')| \leq 2$$, \hspace{1cm} \text{Equation 1}

where

$$E(\alpha, \beta) = \frac{N(\alpha, \beta) + N(\alpha \perp, \beta \perp) - N(\alpha, \beta \perp) - N(\alpha \perp, \beta)}{N(\alpha, \beta) + N(\alpha \perp, \beta \perp) + N(\alpha, \beta \perp) + N(\alpha \perp, \beta)}$$ \hspace{1cm} \text{Equation 2}

$N(\alpha, \beta)$ is the coincidence count for polarizer angles ‘$\alpha$’ and ‘$\beta$’. In the ideal case, the coincidence count is given by:

$$N(\alpha, \beta) = \frac{A}{2} \cos^2(\alpha - \beta)$$ \hspace{1cm} \text{Equation 3}

Where $A$ is the total number of entangled pairs produced.

As indicated by the equations, we violate Bell’s inequality, verifying quantum entanglement, when $|S|$ is greater than 2.

**Experiment**

Due to the very small percentage that go through spontaneous parametric down conversion, a powerful laser must be used. We used a laser of 25.1 mW and a wavelength of 405.5nm. The light passes through a blue filter, and then through a quartz plate. Light is then reflected off a mirror and directed through the two BBO crystals. A beam stop is placed after the BBO crystals, on the axis of the laser, for light that doesn’t go through spontaneous parametric down conversion. The APDs are symmetrically placed at an angle from the laser beam axis such that they are on the axes of intersection of the cones of entangled light. Polarizers are placed in between the two BBO crystals and the APDs. The experimental setup is shown in Figure 3.
Procedure:

1) **Ensuring that all lights are off**, turn on the EM-CCD to observe the cone of down-converted light, using attenuating filters.

2) **Ensuring that all lights are off**, turn on the APDs. Due to the power of the laser, make sure to not have eyes on the same plane as the setup, as the laser could damage eyes.

3) Measure coincidence counts for various angles of the quartz plate.

4) Keeping one polarizer angle constant, alter the angle of the other, taking coincidence count measurements for each angle. Plot coincidence counts vs. angle.

5) To calculate Bell’s inequality, measure coincidence counts for all possible combinations (16) of polarizer angles for polarizer a \{-45, 0, 45, 90\} and polarizer b \{-22.5, 22.5, 67.5, 112.5\}. Calculate Bell’s inequality

Results

To begin, we first measured the single counts of the two APD’s while keeping one polarizer fixed and rotating the second.

![Figure 4: Singles Count Rate for One Fixed Polarizer](image)

For (a), Polarizer A is fixed at 135°. For (b), Polarizer B is fixed at 45°. Count rates between the two detectors seem to follow the same behavior, showing general downwards trends and bumps in the same ranges.
As shown in the graphs, the single count rates for the two detectors do not depend on angle. We expected this, because the incoming light is unpolarized, and therefore the single count rate should not be affected by the angle of the polarizers.

In order to show that the photons entering the detectors are entangled, we must examine the coincidence counts between the two detectors. In our experiment, two measurements are considered simultaneous if they are 26ns apart. The computer software measures the average coincidence rate \( N_{\text{avg}} \) for the two detectors at each angle. However, in order to get the most accurate results, we must take into account the number of accidental coincidences. The accidental coincidence rate for a specific pair of angles is given by:

\[
N_{ac} = N_A N_B \frac{26\text{ns}}{\tau}
\]  
Equation 4

Where \( N_A \) and \( N_B \) are the single count rates of detectors A and B, respectively, and \( \tau \) is the acquisition time window. For our experiment, \( \tau = 0.5s \).

Now we can find the net coincidence count for each angle pair, given by:

\[
N_{\text{net}} = N_{\text{avg}} - N_{ac}
\]  
Equation 5

As shown in Figure 5, the two curves both strongly resemble \( \cos^2 \) curves, which is what we expected from Equation 3. We see that for the blue curve, where Polarizer A is fixed at 135º, the coincidence count maxes when Polarizer B is at 130º and 310º. Similarly, the minima for the curve occur when Polarizer B is at 40º and 220º. The same pattern, although inverted, occurs for the pink curve. From these observations, we can conclude that fixing the angle of one polarizer changes the polarization of the photons hitting the other polarizer, resulting in a maximum when
the angles of the two polarizers are 0° or 180° apart, and a minimum when the angles are 90° apart. This demonstrates how the photons are entangled with respect to polarization, as the angle of one of the polarizers would never affect the coincidence rate under classical conditions.

However, in order to verify that the photons are entangled, the fringe visibility must be greater than 0.71. Fringe visibility is defined as:

\[ V = \frac{N_{\text{max}} - N_{\text{min}}}{N_{\text{max}} + N_{\text{min}}} \]  

Equation 6

The fringe visibility for the two curves are \( V = 0.943 \) and \( V = 0.938 \) for Polarizer A = 135° and Polarizer B = 45°, respectively. Therefore, since the fringe visibility exceeds 0.71, we can confirm that the photons are entangled, and may proceed to try to violate Bell’s Inequalities.

Next, we wish to find the horizontal and vertical angles of the quartz plate that result in the same average coincidence count for four polarization angles. This will give us the optimal orientation of the quartz plate for measuring coincidence counts to find Bell’s Inequality.

![Figure 6: Coincidence Counts vs. Quartz Plate Angles](image)

Coincidence counts for when both polarizers are set to the same angle. For (a), the vertical angle was fixed at 0°. For (b), the horizontal angle was fixed at 4°. Ideally, all four lines on both graphs should have intersected at the same point.

The four lines of Figure 6(a) and (b) should have all intersected at one point. However, since they did not, the near-intersection of the (0,0), (90,90), and (135,135) lines at 4° in 6(a) was used as the ideal horizontal angle for 6(b). The near-intersection at 2° in 6(b) was determined to be the best vertical angle.

Finally, we measured coincidence counts at 16 different angle pairs to calculate Bell’s Inequality from Equations 1 and 2. If \( S \geq 2 \), then Bell’s Inequality has been violated and this demonstrates non-classical behavior. We predict that we should violate Bell’s Inequality at these angles, as \( S \) can be calculated with Equation 1 to have a maximum value of 2.828.
<table>
<thead>
<tr>
<th>Polarizer A angle (°)</th>
<th>Polarizer B angle (°)</th>
<th>Net coincidence</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>α = -45</td>
<td>β = 22.5</td>
<td>16.429 ± 6.487</td>
<td>E(α,β) = .592</td>
</tr>
<tr>
<td></td>
<td>β’ = 22.5</td>
<td>1.240 ± 1.626</td>
<td></td>
</tr>
<tr>
<td></td>
<td>β₁ = 67.5</td>
<td>4.984 ± 1.918</td>
<td></td>
</tr>
<tr>
<td></td>
<td>β’₁ = 112.5</td>
<td>19.577 ± 4.006</td>
<td></td>
</tr>
<tr>
<td>α’ = 0</td>
<td>β = 22.5</td>
<td>15.768 ± 4.097</td>
<td>E(α’,β’) = .059</td>
</tr>
<tr>
<td></td>
<td>β’ = 22.5</td>
<td>27.463 ± 4.154</td>
<td></td>
</tr>
<tr>
<td></td>
<td>β₁ = 67.5</td>
<td>26.967 ± 5.600</td>
<td></td>
</tr>
<tr>
<td></td>
<td>β’₁ = 112.5</td>
<td>14.542 ± 3.419</td>
<td></td>
</tr>
<tr>
<td>α₁ = 45</td>
<td>β = 22.5</td>
<td>48.121 ± 7.060</td>
<td>E(α’,β) = .050</td>
</tr>
<tr>
<td></td>
<td>β’ = 22.5</td>
<td>43.350 ± 8.375</td>
<td></td>
</tr>
<tr>
<td></td>
<td>β₁ = 67.5</td>
<td>6.732 ± 3.342</td>
<td></td>
</tr>
<tr>
<td></td>
<td>β’₁ = 112.5</td>
<td>12.535 ± 2.965</td>
<td></td>
</tr>
<tr>
<td>α’₁ = 90</td>
<td>β = 22.5</td>
<td>10.423 ± 3.267</td>
<td>E(α,β’) = -.789</td>
</tr>
<tr>
<td></td>
<td>β’ = 22.5</td>
<td>21.028 ± 5.813</td>
<td></td>
</tr>
<tr>
<td></td>
<td>β₁ = 67.5</td>
<td>25.539 ± 4.574</td>
<td></td>
</tr>
<tr>
<td></td>
<td>β’₁ = 112.5</td>
<td>12.535 ± 2.965</td>
<td></td>
</tr>
</tbody>
</table>

**S = 1.490 ± .203**

Figure 7: Bell’s Inequality Measurements
Quartz plate horizontal angle at 4° and vertical angle at 2°. E calculated from Equation 2. S calculated from Equation 1. Bell’s Inequality was not violated.

As shown in Figure 7, S was calculated to be 1.490 ± .203, which does not violate Bell’s Inequality. We tried again to violate Bell’s Inequality for the same set of angles and for two different sets of angles that we chose arbitrarily.

<table>
<thead>
<tr>
<th>Polarizer A angles (°)</th>
<th>Polarizer B angles (°)</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>α = -45</td>
<td>β = -22.5</td>
<td>S = 2.691 ± .113</td>
</tr>
<tr>
<td>α’ = 0</td>
<td>β’ = 22.5</td>
<td></td>
</tr>
<tr>
<td>α = 0</td>
<td>β = 22.5</td>
<td>S = 2.560 ± .128</td>
</tr>
<tr>
<td>α’ = 45</td>
<td>β’ = 67.5</td>
<td></td>
</tr>
<tr>
<td>α = -25</td>
<td>β = -2.5</td>
<td>S = 2.318 ± .115</td>
</tr>
<tr>
<td>α’ = 20</td>
<td>β’ = 42.5</td>
<td></td>
</tr>
</tbody>
</table>

Figure 8: Bell’s Inequality for Arbitrary Angles
Bell’s Inequality calculated at angles other than the expected case, with the same method as with Figure 7. All chosen angles were shown to violate Bell’s Inequality.

This time, as all calculated S values are greater than 2, we were able to violate Bell’s Inequality for the expected case and two arbitrary cases. This shows that the photons are not following classical behavior, and therefore verifies that the photons hitting the detectors are entangled.

**Conclusion**

From the experiment, we were able to verify that we had produced polarization-entangled photons. The coincidence count of the two APDs while keeping one polarizer fixed and rotating the other yielded a cosine squared curve. A cosine squared curve is the result when linearly polarized light passes through a rotating polarizer. Although the incident light in the path of the
rotated polarizer was unpolarized when it left the crystals, it became polarized because of the angle of the fixed polarizer. The photons hitting the fixed polarizer caused their entangled partners in the other path to change polarization, resulting in the cosine squared behavior that would normally only occur from polarized light. The fringe visibility of over 0.71 verified that the photons were entangled.

Next, we calculated Bell’s Inequality with multiple sets of polarizer angles. In our first trial, we did not violate Bell’s Inequality \((S = 1.490 \pm 0.203)\) despite choosing ideal angles that should have yielded the maximum value for \(S\). This may have been due to the quartz plate not being at the ideal horizontal or vertical angle or due a misalignment in the system. In any case, because 2.828 is the maximum value of \(S\) at these angles, it is never guaranteed that we will exceed 2 at all. For the later trials, we were able to violate Bell’s Inequality for each set of angles, including the arbitrary angles. Because at some angles entangled photons follow the classical limit of Bell’s Inequality, we might not have violated it for our arbitrary angles had we randomly picked different sets. However, as discussed earlier, failing to violate Bell’s Inequality does not automatically indicate classical behavior. Therefore, though we verified with our chosen angles that the photons were entangled, had we chosen different angles and failed to violate Bell’s Inequality, the photons still would have been entangled.

**Student Contributions**

Kara wrote the Results and Conclusion sections. Ben wrote the Abstract, Theory and Background, and Experiment sections.

**References**

OPT253 Lab 1 Manual