How big is a quantum computer?

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Accounting for resources is the central issue in computational efficiency. We point out physical constraints implicit in information readout that have been overlooked in classical computing. The basic particle-counting mode of read-out sets a lower bound on the resources needed to implement a quantum computer. As a consequence, computers based on classical waves are as efficient as those based on single quantum particles.

Every quantum information processor must be coupled to a decoder, a device that condenses the contents of the quantum register into the classical information that makes up the computational output. The details of the decoder itself can have a profound effect on the power of a given quantum information processor. In the laboratory, decoding is usually done by particle counting, using a detector with finite spatial, temporal, spectral, etc., resolution – and this has some significant implications for the resources that may be needed to implement a quantum processor.

The formal theory of processor operation (the quantum state-space algebra) does not sharply distinguish between a quantum information processor consisting of a few particles occupying many states or many particles occupying a few states. But obviously the decoder (particle counter) can make the distinction easily. The distinction is closely related to the different kinds of entanglement central to advantages accrued by working in the quantum domain. For a single particle, entanglement is usually defined in terms of states belonging with different degrees of freedom of the particle (and associated with commuting observables). For many-particle systems entanglement is possible in addition between identical degrees of freedom of different particles. In any case a particle-counting register readout simply provides a dichotomic answer to the question as to whether there was a particle in a particular mode of the system as a whole.

Information processing by photons provides an illustration. A typical apparatus contains photodetectors whose function is to count by photoionization the number of photons in a certain spatial and spectral range. The photon-counting detector can therefore be understood to give a direct answer to the question: How many photons are in detector mode \( \lambda \)? Note that the mode label \( \lambda \) refers to a set of numbers specifying the modal character. For a photon, these might be the frequency, polarization and wavevector. A different and equally uncomplicated example is provided by the measurement of the electronic state of a Rydberg atom. Standard ramped-field ionization detection gives an answer to the question: Is a particular electronic state, representing an eigenmode of the Schrödinger equation, occupied or not? In fact, because of the particle-counting register readout, a quantum information processor that is implemented by means of the modal entanglement of a single particle can be implemented with equal efficiency by classical wave interference. It may be that detection schemes that do not rely on particle counting could show improvements beyond those of classical wave interferometers. However, we are aware of no laboratory schemes for implementing such measurements.

Any von-Neumann type measurement requires for each of \( N \) eigenvalues of interest a distinct measurement capability. In the simplest example one can envisage these might be the \( N \) channels in a Zeeman or Stern-Gerlach apparatus. For such a case, one says that a "resource" of order \( N \) must be on hand. With this notion of a measurement and its resources, we can quantify the "resource space" needed to implement a combination of processor and decoder. It is simply the total spatio-temporal volume of the modes defining the measurement capabilities. This aspect of the resource requirement is usually ignored in classical computer science, because it is assumed that such a volume may be infinitesimal. Yet this is completely unphysical: quantum mechanics requires that the phase-space volume associated with each mode must be at least \( \hbar \), and consequently the spatio-temporal volume of a mode cannot be zero for a system with finite energy. In addition to counting the resources required for the processor, quantum mechanics therefore forces us to reassess the resources required for readout.

In the specific case of a single-particle processor, the quantum register that is processed during the computation is expressed by a quantum state in which the particle occupies a superposition of \( N \) different modes. The readout of such a register then requires at least \( \log_2 N \) distinct detectors or generalized resources. The classical outputs of these detectors then have to be polled to locate the sequence of detectors that fired. This is simply the classical search problem for a sorted database, and it is evident that such a search can be done most efficiently in \( \log_2 N \) steps by means of a binary search. This method of resource counting can be adopted to show that any single-particle processor can be efficiently implemented using classical wave interference, even when the system relies exclusively on the entanglement of different degrees of freedom of the particle.
A description of the quantum processor in terms of the system state vector alone makes it difficult to distinguish single-particle from many-particle situations, let alone to understand the optimal combinations of particles and degrees of freedom that might be needed to accomplish a certain processing task. The essential roles of interference and entanglement can be examined unambiguously most simply in the language of second quantization. The field operator \(\hat{\psi}(x)\) describes the annihilation of atoms, photons or other types of particles at position \(x\). If we introduce a complete set of orthonormal spatial modes \(u_\lambda(x)\), where \(\lambda\) labels the different modes including the possibility of having several degrees of freedom, we may introduce mode annihilation operators \(\hat{a}_\lambda\) via \(\hat{\psi}(x) = \sum_\lambda \hat{a}_\lambda u_\lambda(x)\), where \([\hat{a}_\lambda, \hat{a}^\dagger_\lambda] = \delta_{\lambda\mu}\) depending on whether the field describes fermions or bosons.

If the unitary operator \(\hat{U}\) defines the evolution of a quantum processor, then the output register is given by

\[
\hat{\psi}(x, t) = \hat{U}(t) \hat{\psi}(x, 0) \hat{U}^\dagger(t),
\]

where \(\hat{\psi}(x, 0)\) represents the input register. The only restriction on the Hamiltonian generating \(\hat{U}\) is that it preserves the particle number of the processor register: If this register contains only a single particle, so that the state is given by \(|\psi\rangle = \sum_\lambda \psi_\lambda \hat{a}^\dagger_\lambda \text{vac}\rangle\), the only terms in the normally ordered Hamiltonian contributing to the dynamical evolution are those bilinear in \(\hat{a}_\lambda^\dagger\) and \(\hat{a}_\lambda\). Thus the unitary evolution \(\hat{U}\) reduces from the operators to the mode functions themselves:

\[
\hat{\psi}(x, t) = \sum_\lambda \hat{a}_\lambda u_\lambda(x, t),
\]

where \(u_\lambda(x, t)\) are the time-propagated mode functions of wave mechanics.

The operation of an elementary measurement can be formulated in the language of fields as the action of a linear filter on the underlying mode structure of the system. The register field operator after the filter is given by

\[
\hat{\psi}_\mu(x, t) = \int dx' \int dt' \Gamma_{\mu}(x, t| x', t') \hat{\psi}(x', t'),
\]

where \(\mu\) denotes the mode that is to be probed for particles. The observable corresponding to an elementary measurement is given by the number operator for this mode

\[
\hat{N}_\mu = \int dx \int dt \hat{\psi}_\mu^\dagger(x, t) \hat{\psi}_\mu(x, t).
\]

In each measurement one determines the number of particles in the specified detector mode. This can be described by the projector \(|N;\mu\rangle \langle N;\mu| = \delta(\hat{N}_\mu - N)\), where \(|N;\mu\rangle\) denotes the state with \(N\) particles in the detected mode \(\mu\) and \(\delta(N)\) is understood as the Kronecker delta function. In the case of a single particle, \(N\) may take the values \((0, 1)\), so that the projector simplifies to

\[
\delta(\hat{N}_\mu - N) = \delta(N) (\hat{1}_\mu - \hat{N}_\mu) + \delta(N-1) \hat{N}_\mu,
\]

showing that the measurement projector can be entirely expressed in terms of the bilinear number operator (8).

As indicated previously, the minimum number of detectors required to read out the \(N\)-channel register is \(\log_2 N\). This can be achieved for any quantum system by representing each of the mode labels \(\mu\) by a set of \(\log_2 N\) bits \(\{b_i\}\) and grouping the number operators of the modes, \(\hat{N}_{(b_i)}\), into a specific set of detectable number operators \(M_\alpha\) according to their common bits. For example, in a system of \(N = 8\) different modes (i.e. with a Hilbert space of eight dimensions) the three number operators are

\[
M_{1xx} = \hat{N}_{(1,0,0)} + \hat{N}_{(1,1,0)} + \hat{N}_{(1,1,1)} + \hat{N}_{(1,1,1)}, \quad (6)
\]

\[
M_{2xx} = \hat{N}_{(0,1,0)} + \hat{N}_{(0,1,1)} + \hat{N}_{(1,1,0)} + \hat{N}_{(1,1,1)}, \quad (7)
\]

\[
M_{xx1} = \hat{N}_{(0,0,1)} + \hat{N}_{(0,1,1)} + \hat{N}_{(1,0,1)} + \hat{N}_{(1,1,1)}. \quad (8)
\]

The readout would then occur via a cascaded sequence of particle counters. The result of measuring the three observables (6) is the set of bits specifying the mode. Here too the projectors corresponding to the operators \(M_\alpha\) take the form (6). Note that it is not necessary that each of the bits correspond to a different degree of freedom of the particle, i.e. \(\mu = \{\mu_1, \mu_2, \ldots\}\). The binary readout scheme is possible even for a single particle excited in one degree of freedom. In practice it may prove simpler to use different degrees of freedom (such as the principle quantum number, and the two angular momentum quantum numbers for a Rydberg electron), though no such scheme has been analyzed in detail as yet. Though the number of detectors is minimal, the number of resources in terms of space-time volume of required modes is

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**Figure 1** Elements of a quantum computer. The encoder transcribes the classical input into the quantum register of the processor. After completion of the processing, the quantum register is read out by the decoder to obtain the classical computational output.
not, as can be seen from the fact that each measured number
orbit involves more than one mode.

If the result of the computation leaves the register in one of
the detected modes, then the readout provides directly a
classical result for the computation. In general, however, the
computation will place the register in a superposition state, so
that the readout will be probabilistic. That is, for many runs of
the same computation the classical outputs will be different.

Therefore the whole computational process has to be repeated
many times, either sequentially or in parallel. Thus in both
cases one must determine for each mode the expectation value
of the projector (5).

For a single particle it is straightforward to show using
Eqs. (6) – (8) that this expectation value is proportional to the
correlation function

\[ C(x, t; x', t') = \langle \hat{\psi}^\dagger(x, t) \hat{\psi}(x', t') \rangle. \]

Using the general single-particle state \( |\psi\rangle = \sum \psi_\lambda a_\lambda^\dagger |\text{vac}\rangle \)
and the dynamics according to Eq. (9) we obtain

\[ C(x, t; x', t') = \psi^\dagger(x, t) \psi(x', t'), \]

where we have defined the single-particle wavefunction as
\( \psi(x, t) = \sum \psi_\lambda u_\lambda(x, t) \). This result holds for both the
single-atom and single-photon case and is equivalent to a
classical interference pattern obtained by replacing the
expectation value (10) by the classical electric-field correlation
function

\[ C_{\text{class}}(x, t; x', t') = E^\dagger(x, t) E(x', t'). \]

where \( E(x, t) \) is an analytic signal. A comparison of Eqs (10)
and (11) shows a complete one-to-one correspondence be-
tween classical interference and single-particle quantum in-
terference, not only in the dynamics but also in the mea-
surement. We conclude therefore that any quantum computer
based on a single-particle quantum register can be im-
plemented with equal efficiency entirely by a classical inter-
ferometer. This is because the concept of the entanglement of
degrees of freedom of a single particle cannot be attributed an
inherently quantum character: it is perfectly understandable
in terms of classical wave interference.

This formulation also provides some insight into the issue
of the speedup of a particular algorithm using quantum interfer-
ence. Consider, for example, a quantum search algorithm
that is based on single-particle entanglement, that is, on
the correlation of different degrees of freedom of a single particle
at the level of probability amplitudes. If this case the quan-
tum register may be realized by \( N \gg 1 \) modes but the modes
only be occupied by a total population of 1. In the optical
case, these might be the modes associated with the polariza-
tion and the wavevector degrees of freedom. Alternatively
the energy and angular momentum degrees of freedom of a
Rydberg electron may be used.

The readout requires \( O(N) \) resources and at minimum
\( \log_2 N \) detectors, whose classical outputs can be searched in
not less than \( O(\log_2 N) \) steps. In fact, it is evident that a
classical search is always required in the final readout, since
the values of the output quantum register only obtain physi-
cal reality after having been measured. Therefore, any quan-
tum computation with a single particle cannot be faster than
\( O(\log_2 N) \). Moreover, by virtue of the one-to-one corre-

dence described above, it can be performed using classical
interference with the same speed and the same number of
resources. This is in striking contrast to the belief that a
quantum-search algorithm, even without using multi-particle
entanglement, i.e. with only a single particle, can be advan-
tageous for solving the search problem.

The quantum computation itself consists of preparing the
input register and performing a unitary \( N \times N \) transformation
\( \hat{U} \). According to Grover, this unitary transformation may
perhaps consist of a series of operations like passing the sin-
gle particle through a so-called Oracle, whose function is to
alter the phase of one of the modes, followed by an \( N \times N \)-port
beamsplitter. These steps are usually referred to as “querying
the Oracle” and “inversion about the mean”. Using these op-
erations it has been shown that an unsorted database can be
searched by performing only \( O(\sqrt{N}) \) “queries” or even only
a single “query” of the Oracle. This is in stark contrast to
the corresponding classical search which requires \( O(N) \)
queries.

The caveat, as Steane has pointed out, is that one should
be careful of comparing an inefficient classical algorithm with
an efficient quantum one. The experimental implementations
of the Grover search to date have not in fact implemented
a search of an unsorted database, but rather a database with
one single marked item. Therefore experimental evidence for
the speedup has yet not been achieved, since a sorted
database can be searched classically in \( \log_2 N \) steps, which is
identical to the readout limit for a quantum computer.

The superiority of the quantum search algorithm becomes
apparent only when one examines carefully the notion of a
query. It is evident that there is no information gain in the sort
of oracle query described above, which involves only unitary
transformations. It is in the encoding of such oracles, which
must be done by a quantum computer, that the real speedup
occurs. Once a properly encoded oracle is available, the num-
er of steps required to perform the information processing is
then limited by the final register readout to \( O(\log_2 N) \). This
is exactly the same as for a classical information processor,
which shows that it is the realization of actual information
by the readout, as opposed to predictive information that is
contained in the quantum state, that is the ultimate limit-
ing procedure in quantum information processing.

We have shown here that if the readout of the register is
performed by particle counting then there exists a one-to-one
correspondence of single-particle quantum interference and
classical interference. Therefore we conclude that any enhancements in processing power that can be ascribed to quantum interference can also be found in classical wave processors, and this includes systems based on modal entanglement. Multi-particle entanglement, on the other hand, may provide enhancements that cannot be efficiently transcribed to classical interferometers, even when particle counting is used to realize the output information.

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